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On Bernstein–Szegö Orthogonal Polynomials on Several Intervals. II. Orthogonal Polynomials with Periodic Recurrence Coefficients

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Geronimus has shown that a sequence of orthogonal polynomials (p_n) with periodic recurrence coefficients for $n \geqslant n_0$ is orthogonal on a set of disjoint intervals $E_l = \bigcup_{j=1}^{l} [a_{2j-1}, a_{2j}]$ with respect to a distribution of the form

$$d\psi(x) = \sqrt{-\prod_{j=1}^{2l} (x-a_j)}/|\rho_\nu(x)|\, dx + d\mu(x),$$

where $\rho_\nu(x) = \prod_{j=1}^{\nu} (x - w_j)$ with $\operatorname{sgn} \rho_\nu(x) = (-1)^{l+1-j}$ on (a_{2j-1}, a_{2j}) for $j = 1, ..., l$, $\nu \geqslant l-1$, and where μ is a certain point measure with $\operatorname{supp}(\mu) \subset \{w_1, ..., w_\nu\}$. In this paper we show (in fact a more general result is presented) that a sequence of polynomials (p_n) orthogonal with respect to $d\psi$ has recurrence coefficients of period N, $N \geqslant l$, for $n \geqslant n_0$, if and only if there exists a so-called Chebyshev polynomial $\mathscr{T}_N$ of degree N on E_l, where a polynomial $\mathscr{T}_N$ is called a Chebyshev polynomial on E_l if $|\mathscr{T}_N|$ attains its maximum value on E_l at $N+l$ points from E_l. Furthermore it is demonstrated how to get in a simple way a (nonlinear) recurrence relation for the recurrence coefficients of the orthogonal polynomials. Results on Chebyshev polynomials on several intervals are also given. © 1991 Academic Press, Inc.

1. INTRODUCTION AND NOTATION

First let us note that the notation will be the same as in [16] which will be referred to as I. The references to the equalities and sections in I will be made by prefix I, e.g., (I.3.1) means equality (3.1) in I. On the other hand, (3.1) means equality (3.1) of this paper.

Henceforth let $\mathbf{N} = \{1, 2, 3, ...\}$ and $\mathbf{N}_0 = \{0, 1, 2, ...\}$, $l \in \mathbf{N}$, $a_k \in \mathbf{R}$ for $k = 1, ..., 2l$, $a_1 < a_2 < \cdots < a_{2l}$ and put

$$E_l = \bigcup_{k=1}^{l} [a_{2k-1}, a_{2k}], \qquad H(x) = \prod_{k=1}^{2l} (x - a_k)$$

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and

$$1/h(x)=\begin{cases}(-1)^{l-j}/\pi\sqrt{|H(x)|} & \text{for} \quad x\in(a_{2j-1},a_{2j}),\ j=1,...,l,\\ 0 & \text{elsewhere.}\end{cases} \tag{1.1}$$

R and S are real polynomials with leading coefficient one and $\partial R=r$ and $\partial S=s$ $(r+s=2l)$ which satisfy the relation

$$R(x)\,S(x)=H(x).$$

As usual, ∂p denotes the exact degree of the polynomial p, ρ_ν denotes a real polynomial with $\partial\rho_\nu=\nu$ which has no zero in E_l, i.e.,

$$\rho_\nu(x)=c\prod_{k=1}^{\nu^*}(x-w_k)^{\nu_k},$$

where $c\in\mathbf{R}\backslash\{0\}$, $\nu^*\in\mathbf{N}_0$, $\nu_k\in\mathbf{N}$ for $k=1,...,\nu^*$, $\nu=\sum_{k=1}^{\nu^*}\nu_k$, $w_k\in\mathbf{C}\backslash E_l$ for $k=1,...,\nu^*$, and the w_k's are real or appear in pairs of complex conjugate numbers. Furthermore set

$$\rho_{\nu,k}(x)=\rho_\nu(x)/(x-w_k)^{\nu_k} \qquad \text{for} \quad k=1,...,\nu^*.$$

In what follows we choose always that branch of $\sqrt{H}$ which is analytic on $\mathbf{C}\backslash E_l$ and which satisfies

$$\operatorname{sgn}\sqrt{H(y)}=\operatorname{sgn}\prod_{k=1}^{l}(y-a_{2k-1}) \qquad \text{for} \quad y\in\mathbf{R}\backslash E_l. \tag{1.2}$$

For given $R,\rho_\nu,\varepsilon=(\varepsilon_1,...,\varepsilon_{\nu^*})$, $\varepsilon_k\in\{-1,1\}$, let us now define the following linear functionals on the space of real polynomials $\mathbf{P}$

$$L_{R,\rho_\nu,\varepsilon}(p)=\sum_{k=1}^{\nu^*}\frac{(1-\varepsilon_k)}{(\nu_k-1)!}\left(\frac{pR}{\rho_{\nu,k}\sqrt{H}}\right)^{(\nu_k-1)}(w_k) \qquad \text{for} \quad p\in\mathbf{P}, \tag{1.3}$$

and

$$\Psi_{R,\rho_\nu,\varepsilon}(p)=\int p\frac{R}{\rho_\nu h}\,dx+L_{R,\rho_\nu,\varepsilon}(p) \qquad \text{for} \quad p\in\mathbf{P}, \tag{1.4}$$

where it is assumed that $\varepsilon_{k+1}=\varepsilon_k$, if w_k and w_{k+1} are complex conjugate, here $g^{(j)}$ denotes the jth derivative of g. If there is no possible confusion indices ν resp. R,ρ_ν,ε are omitted. The unique sequence of orthogonal polynomials $(p_{i_n})_{n\in\mathbf{N}_0}$, $p_{i_n}=x^{i_n}+\cdots$, $i_0=0$, satisfying

$$\Psi_{R,\rho_\nu,\varepsilon}(x^jp_{i_n})=0 \qquad \text{for} \quad j=0,...,i_{n+1}-2$$

and

$$\Psi_{R,\rho_\nu,\varepsilon}(x^{i_{n+1}-1}p_{i_n}) \neq 0$$

have been investigated by the author in [16]. If $\Psi_{R,\rho_\nu,\varepsilon}$ is definite, i.e., if $i_n = n$ for $n \in \mathbf{N}_0$, then it is well known (see, e.g., [3]) that the polynomials p_n, $n \in \mathbf{N}_0$, satisfy a recurrence relation of the form

$$p_n(x) = (x - \alpha_n)\, p_{n-1}(x) - \lambda_n p_{n-2}(x) \qquad \text{for} \quad n \in \mathbf{N}, \tag{1.5}$$

where $p_{-1} \equiv 0$, $p_0 \equiv 1$, $\alpha_n \in \mathbf{R}$, and $\lambda_{n+1} \in \mathbf{R} \backslash \{0\}$ for $n \in \mathbf{N}$.

In this paper we study orthogonal polynomials with periodic recurrence coefficients, i.e., polynomials p_n which satisfy a recurrence relation of the form (1.5) and the recurrence coefficients of which satisfy the periodic conditions

$$\alpha_{N+n+2} = \alpha_{n+2} \qquad \text{and} \qquad \lambda_{N+n+2} = \lambda_{n+2} \quad \text{for } n \geqslant n_0,$$

where $N \in \mathbf{N}$ and $n_0 \in \mathbf{N}$.

Geronimus [7] (see, e.g., [5]) has shown, under the additional assumption that $\lambda_{n+1} \in \mathbf{R}^+$ for $n \in \mathbf{N}$, that such polynomials are orthogonal with respect to a positive definite linear functional $\Psi_{H,\rho,\varepsilon}$, where $l \leqslant N$ (corresponding results for orthogonal polynomials with asymptotically periodic recurrence coefficients have been given recently by Geronimo and Van Assche [5]). Thus the question arises whether polynomials orthogonal with respect to $\Psi_{H,\rho,\varepsilon}$ have periodic recurrence coefficients. For the case that

$$\Psi_{H,\rho,\varepsilon}(p) = \int_{E_l} p \frac{\sqrt{-H}}{|\rho|}\, dx \qquad \text{for} \quad p \in \mathbf{P},$$

where $\operatorname{sgn} \rho = -\operatorname{sgn} h$ on $\operatorname{int}(E_l)$, A. Magnus [10, Sect. 4.2] has shown, based on results of Nuttall and Singh [12], that the recurrence coefficients have periodic or quasi-periodic behaviour, where this fact is explained by special Abel functions the periods and amplitudes of which depend only on E_l. Examples show (see also [14]) that in general periodicity of the recurrence coefficients can not be expected if E_l consists of more than one interval. In the single interval case it is well known by the results of Bernstein and Szegö [18] that the recurrence coefficients are constant for $n \geqslant \nu$ and thus have period one. In this paper we demonstrate that polynomials which are orthogonal with respect to $\Psi_{R,\rho,\varepsilon}$, $\Psi_{R,\rho,\varepsilon}$ definite, have recurrence coefficients of period N if and only if there exists a Chebyshev polynomial (abbreviated T-polynomial), $\mathscr{T}_N = x^N + \cdots$ on E_l, where a polynomial $\mathscr{T}_N = x^N + \cdots$ is called a T-polynomial on E_l if $|\mathscr{T}_N|$ attains its maximum value at $N + l$ points in E_l. Moreover we have, taking into consideration Geronimus's result, that the existence of a system of orthogonal polynomials with periodic recurrence coefficients and spectrum E_l, up to a finite point spectrum, is equivalent to the existence of a T-polynomial on E_l.

We proceed as follows: In the second section we characterize T-polynomials on disjoint intervals and give some basic properties of such polynomials. In the third section we demonstrate the above mentioned result on the periodicity of the recurrence coefficients. In the fourth section we investigate the connection between the recurrence coefficients of the polynomials orthogonal with respect to $\Psi_{R,\rho,\varepsilon}$ resp. $\Psi_{R,\rho,-\varepsilon}$. Special attention is given to the interesting case where the recurrence coefficients are symmetric periodic. In the fifth section we show how to get in a very simple way recurrence relations for the recurrence coefficients if the period is greater than the number of the intervals. Using completely different methods these recurrence relations for the recurrence coefficients were derived by Turchi *et al.* in [19].

Finally we would like to mention that polynomials with periodic recurrence coefficients appear also in the papers of Kac and Van Moerbeke [9] and Van Moerbeke [11] where periodic Jacobi matrices are investigated and the connection of such matrices resp. of orthogonal polynomials having periodic recurrence coefficients with periodic Toda lattices is demonstrated.

As it was brought to our attention by the referee the question of periodicity resp. asymptotic periodicity of the recurrence coefficients was also investigated by Aptekarev [21, see in particular Sect. 3]. Based on Widom's asymptotic formulas for polynomials orthogonal on a system of contours, see [20], he demonstrated that a sequence of polynomials (p_n) orthogonal with respect to a positive measure μ has asymptotically periodic recurrence coefficients of period N if the spectrum of μ consists of N disjoint intervals $[a_{2j-1}, a_{2j}]$, $j=1, ..., N$, of equal harmonic measure at ∞ (which is equivalent to the fact that there exists a T-polynomial of degree N on $E_N = \bigcup_{j=1}^{N} [a_{2j-1}, a_{2j}]$) and of finitely many discrete values and that the absolutely continuous part of the measure μ satisfies a generalized Szegö condition on E_N. Furthermore a necessary condition which is close to the above stated sufficient condition is given.

2. Chebyshev Polynomials on Disjoint Intervals

Definition 2.1. Let $l, N \in \mathbf{N}$ and suppose that $N \geqslant l$. We call a polynomial $\mathscr{T}_N(x) = x^N + \cdots$, $N \in \mathbf{N}$, a Chebyshev polynomial (abbreviated T-polynomial) on E_l, if $\mathscr{T}_N$ is of the form

$$\mathscr{T}_N^2(x) = H(x)\,\mathscr{U}_{N-l}^2(x) + L^2, \tag{2.1}$$

where $\mathscr{U}_{N-l}$ is a polynomial of degree $N-l$ with leading coefficient one and $L \in \mathbf{R}^+$. $\tilde{\mathscr{T}}_N(x) = \mathscr{T}_N(x)/L$ is called a normed T-polynomial on E_l.

Let us note that $\mathscr{U}_{N-l}$ and L from (2.1) are uniquely determined and that it follows from (2.1), since $H \leqslant (>) 0$ on $E_l(\mathbf{R}\backslash E_l)$, that

$$|\mathscr{T}_N(x)| \begin{cases} \leqslant L & \text{for} \quad x \in E_l, \\ > L & \text{for} \quad x \in \mathbf{R}\backslash E_l. \end{cases}$$

We have chosen the name "T-polynomial" because it turns out in this section that polynomials satisfying (2.1) have a similar behaviour on E_l as the well known Chebyshev polynomials on $[-1, +1]$. T-polynomials on two disjoint intervals have been investigated by Achieser [1] and then by the author [14, 15]. The reason why we are interested in T-polynomials in this paper is, as we shall demonstrate in the next section, that there is an equivalence between the existence of a T-polynomial on E_l and the periodicity of the recurrence coefficients of the polynomials orthogonal with respect to functionals of the type $\Psi_{R,\rho,\varepsilon}$.

First let us characterize T-polynomials with the help of an orthogonality property and let us show how to determine algebraically those disjoint intervals E_l on which there exists a T-polynomial.

THEOREM 2.1. (a) *$\mathscr{T}_N$ is a T-polynomial on E_l if and only if $\mathscr{T}_N \perp \mathbf{P}_{N+l-2}$ on E_l with respect to $1/h$. ($\mathbf{P}_n$ denotes as usual the set of real polynomials of degree at most n.)*

(b) *There exists a T-polynomial $\mathscr{T}_N$ on E_l if and only if*

$$\begin{vmatrix} m_{N+k} & m_{N+k-1} & \cdots & m_k \\ m_{N+k+1} & m_{N+k} & \cdots & m_{k+1} \\ \vdots & \vdots & & \vdots \\ m_{2N+k} & m_{2N+k-1} & \cdots & m_{N+k} \end{vmatrix} = 0 \qquad \text{for} \quad k = 0, ..., l-2,$$

where

$$m_k = \int_{E_l} x^k \frac{dx}{h(x)} \qquad \text{for} \quad k \in \mathbf{N}_0.$$

Proof. (a) Applying Theorem I.1 resp. Theorem I.3 to relation (2.1) part (a) follows.

(b) We give only a sketch of the proof since the methods are similar to those used in [17, pp. 428–429]. Necessity. Put

$$\mathscr{T}_N(x) = \sum_{j=0}^{N} \beta_j x^{N-j}.$$

Then it follows by (a)

$$\int_{E_l} x^k \left(\sum_{j=0}^{N} \beta_j x^{N-j} \right) \frac{dx}{h(x)} = 0 \qquad \text{for} \quad k = 0, ..., N+l-2 \tag{2.2}$$

and hence

$$\sum_{j=0}^{N} \beta_j m_{N+k-j} = 0 \qquad \text{for} \quad k = 0, ..., N+l-2$$

which is the assertion.

Sufficiency. Let for sufficiently small $|x|$

$$\frac{\sum_{j=0}^{N-1} \alpha_j x^j}{\sum_{j=0}^{N} \beta_j x^j} = \sum_{j=0}^{2N-1} m_{l-1+j} x^j + 0(x^{2N}).$$

Since the determinants given in (b) are zero for $k = 0, ..., l-2$ it follows that

$$\alpha_{N-1-k} = 0 \qquad \text{for} \quad k = 0, ..., l-2$$

and thus, recalling that by Lemma I.1(c), $m_j = 0$ for $j = 0, ..., l-2$,

$$\sum_{j=0}^{N} \beta_j m_{N+k-j} = 0 \qquad \text{for} \quad k = 0, ..., N+l-2,$$

which is equivalent to (2.2). Hence, by (a), the theorem is proved. ∎

Notation. As usual let T_n resp. U_n, $n \in \mathbf{N}_0$, denote the Chebyshev polynomials of degree n of first resp. second kind on $[-1, +1]$.

Furthermore, if $\mathscr{T}_N$ is a T-polynomial on E_l, we put for $n \in \mathbf{N}$

$$\tilde{\mathscr{T}}_{nN} = T_n(\tilde{\mathscr{T}}_N) \qquad \text{and} \qquad \mathscr{T}_{nN} = L^n \tilde{\mathscr{T}}_{nN} / 2^{n-1} = x^{nN} + \cdots, \tag{2.3}$$

and

$$\tilde{\mathscr{U}}_{nN-l} = \tilde{\mathscr{U}}_{N-l} U_{n-1}(\tilde{\mathscr{T}}_N) \qquad \text{and} \qquad \mathscr{U}_{nN-l} = L^n \tilde{\mathscr{U}}_{nN-l} / 2^{n-1} = x^{nN-l} + \cdots, \tag{2.4}$$

where $\tilde{\mathscr{U}}_{N-l} = \mathscr{U}_{N-l}/L$ and $\mathscr{U}_{N-l}$ is defined in (2.1).

If t is a polynomial we use also the notation

$$\hat{t}(x) := t(x)/K, \qquad \text{where } K \text{ is the leading coefficient of } t.$$

LEMMA 2.1. *Let $\mathscr{T}_N$ be a T-polynomial on E_l. Then for $n \in \mathbf{N}$*

$$\tilde{\mathscr{T}}_{nN}^2 - H \tilde{\mathscr{U}}_{nN-l}^2 = 1, \tag{2.5}$$

i.e., the polynomials $\mathscr{T}_{nN}$, $n \in \mathbf{N}$, are T-polynomials on E_l.

Proof. Using the well known relation

$$T_n^2(x)-(x^2-1)\,U_{n-1}^2(x)=1$$

we get with the help of (2.1) that

$$[T_n(\tilde{\mathscr{T}}_N)]^2-H[\tilde{\mathscr{U}}_{N-l}U_{n-1}(\tilde{\mathscr{T}}_N)]^2=1$$

which is the assertion. ∎

The following corollary shows that knowing a T-polynomial on E_l we know an infinite subsequence of polynomials orthogonal with respect to distributions of the type $\rho/h\,dx$, $1/\rho h\,dx+$ point measure, $\Psi_{R,\rho_{l-1},\varepsilon}$, etc.

COROLLARY 2.1. *Let $\mathscr{T}_N$ be a T-polynomial on E_l and let ρ be a polynomial of degree at most $l-1$ which has no zero in* $\operatorname{int}(E_l)$. *Then the following propositions hold*:

(a) *For each $n\in\mathbf{N}$, $\mathscr{T}_{nN}\perp\mathbf{P}_{nN+l-\partial\rho-2}$ on E_l with respect to ρ/h.*

(b) *For each $n\in\mathbf{N}$, $\mathscr{U}_{nN-l}\perp\mathbf{P}_{nN-\partial\rho-2}$ on E_l with respect to ρh.*

(c) *Let σ be a point measure such that the support of σ is a subset of the set of zeros of ρ. Then for each $n\in\mathbf{N}$, $\rho\mathscr{T}_{nN}\perp\mathbf{P}_{nN+l-2}$ on E_l with respect to $1/h\rho\,dx+d\sigma$.*

(d) *Let σ as in* (c). *Then for each $n\in\mathbf{N}$, $\rho\mathscr{U}_{nN-l}\perp\mathbf{P}_{nN-2}$ on E_l with respect to $h/\rho\,dx+d\sigma$.*

Proof. Since by (2.5) and Theorem I.1(a) resp. Theorem I.1(b), $\mathscr{T}_{nN}\perp\mathbf{P}_{nN+l-2}$ resp. $\mathscr{U}_{nN-l}\perp\mathbf{P}_{nN-2}$ with respect to $1/h$ resp. h on E_l the corollary follows immediately. ∎

Remark 2.1. As we have learned quite recently orthogonal polynomials (p_n) with the property that $p_{nN}=r_n(\tilde{\mathscr{T}}_N)$ and $p_{nN-1}^{(1)}=\rho_{l-1}r_{n-1}^{(1)}(\tilde{\mathscr{T}}_N)$ for all $n\in\mathbf{N}$, where (r_n) is a sequence of orthogonal polynomials the spectrum of which is contained in $[-1,+1]$, $r_{n-1}^{(1)}$ denotes the associated polynomials of order one, and ρ_{l-1} is a polynomial of degree $l-1$ which has exactly one zero in $[a_{2j},a_{2j+1}]$, $j=1,\dots,l-1$, have been studied in [6]. The normed T-polynomial $\tilde{\mathscr{T}}_N$ on E_l is called there polynomial mapping. But let us note that in general polynomials orthogonal with respect to $\Psi_{R,\rho_{l-1},\varepsilon}$ resp. with respect to the more general functional $\Psi_{R,\rho,\varepsilon}$ do not fit into this class of orthogonal polynomials because of the following results: Let $\mathscr{T}_N$ be a T-polynomial on E_l and let $\rho_{l-1}(x)=\prod_{k=1}^{l-1}(x-w_k)$ be such that $w_k\in[a_{2k},a_{2k+1}]$ for $k=1,\dots,l-1$. Furthermore let (p_n) be a sequence of orthogonal polynomials with the property that $p_{nN}=r_n(\tilde{\mathscr{T}}_N)$ and $p_{nN-1}^{(1)}=\rho_{l-1}U_{n-1}(\tilde{\mathscr{T}}_N)=\rho_{l-1}\mathscr{U}_{nN-l}$ for all $n\in\mathbf{N}$, where r_n is such that $r_{n-1}^{(1)}=$

$\hat{U}_{n-1}$ for $n \in \mathbf{N}$. Then one gets from [6, (2.17) and (2.14)] or by direct methods that $(p_n^{(1)})$ is orthogonal with respect to $\sqrt{-H/\pi}\,|\rho_{l-1}|\,dx + \sum_{k=1}^{l-1} \mu_k\, \delta(x - w_k)$ on E_l, where the point measure μ_k at w_k is, up to very special cases, different from that one defined in (1.3). More precisely it can be demonstrated that equality of the two point measures can only occur if $\tilde{\mathcal{T}}_N(w_{l-1-2j}) = c_1$ for $j = 0, 1, ..., [(l-1)/2]$ and $\tilde{\mathcal{T}}_N(w_{l-2j}) = -c_2$ for $j = 1, 2, ..., [(l-1)/2]$ where $c_1, c_2 \in [1, \infty)$. For example, if $r_n = \hat{T}_n, n \in \mathbf{N}$, then $\mu_k = -\sqrt{H(w_k)}/\rho'_{l-1}(w_k)$ and thus the point measure differs from that one given in (1.3) by the factor 2. Hence only in the case where ρ_{l-1} has all zeros at boundary points of E_l, i.e., $\sqrt{H(w_k)} = 0$, for $k = 1, ..., l-1$, one obtains a distribution of the type treated in this paper (see Section 4), namely the distribution $\sqrt{-R(x)/S(x)}\,dx$ where $\partial R = l + 1(l-1)$ and $\partial S = l - 1(l+1)$. Finally let us mention that distributions of the form $\sqrt{-R(x)/S(x)}/\rho(\tilde{\mathcal{T}}_N(x))\,dx$, where ρ is a polynomial which is positive on $[-1, +1]$, are the only other distributions which fit also into that class of orthogonal polynomials investigated in [6, compare Remark 7].

Corollary 2.2. *Let $\mathcal{T}_N$ be a T-polynomial on E_l. Then*

(a) *$\mathcal{T}_{nN}$ resp. $\mathcal{U}_{nN-l}$, $n \in \mathbf{N}$, has nN resp. $nN - l$ simple zeros in* $\operatorname{int}(E_l)$.

(b) *There is a unique $r_{l-1} \in \mathbf{P}_{l-1}$ which has exactly one zero in each interval (a_{2j}, a_{2j+1}), $j = 1, ..., l-1$, such that for each $n \in \mathbf{N}$*

$$\mathcal{T}'_{nN} = nNr_{l-1}\mathcal{U}_{nN-l} \qquad \text{and} \qquad 2nNr_{l-1}\mathcal{T}_{nN} = 2H\mathcal{U}'_{nN-l} + H'\mathcal{U}_{nN-l}.$$

(c) *For $n \in \mathbf{N}$*

$$\mathcal{U}_{nN-l}(z) = \int_{E_l} \frac{\mathcal{T}_{nN}(z) - \mathcal{T}_{nN}(x)}{z - x} \frac{dx}{h(x)}$$

and

$$\mathcal{T}_{nN}(z) = \int_{E_l} \frac{(H\mathcal{U}_{nN-l})(z) - (H\mathcal{U}_{nN-l})(x)}{z - x} \frac{dx}{h(x)}.$$

Proof. (a) Follows immediately from Theorem 1 and Lemma I.5.

(b) Let $n = 1$. Since, by (a), $\mathcal{U}_{N-l}$ has all zeros in $\operatorname{int}(E_l)$ and since, by (2.1), $\mathcal{T}_N$ has a local extremum at the zeros of $\mathcal{U}_{N-l}$ it follows that

$$\mathcal{T}'_N = Nr_{l-1}\mathcal{U}_{N-l}, \qquad \text{where} \quad r_{l-1} \in \mathbf{P}_{l-1}.$$

Observing that

$$\mathcal{T}_N(a_{2j}) = \mathcal{T}_N(a_{2j+1}) \qquad \text{for} \quad j = 1, ..., l-1,$$

we get that $\mathcal{T}'_N$ and thus r_{l-1} has at least one zero in each interval (a_{2j}, a_{2j+1}), $j = 1, ..., l-1$, which gives the first relation for $n = 1$. Differentiating $\mathcal{T}_{nN}$ it follows immediately that the first relation holds for each $n \in \mathbf{N}$.

The second relation follows by differentiating (2.5) and using the first relation.

(c) Follows immediately from Theorem I.1. ∎

THEOREM 2.2. (a) *$\mathcal{T}_N = x^N + \cdots$ is a T-polynomial on E_l if and only if there exist exactly $N+l$ points $y_j \in E_l$, $y_1 < y_2 < \cdots < y_{N+l}$, such that $|\mathcal{T}_N(y_j)| = \max_{x \in E_l} |\mathcal{T}_N(x)|$ for $j = 1, ..., N+l$.*

(b) *Each polynomial $\mathcal{T}_N = x^N + \cdots$ with N simple real zeros is a T-polynomial on the set of intervals $E(\mu) = \{x \in \mathbf{R} : |\mathcal{T}_N(x)| \leqslant \mu\}$, where $\mu \in (0, K]$ and $K := \min\{|\mathcal{T}_N(x)| : \mathcal{T}'_N(x) = 0\}$.*

(c) *Suppose that $\mathcal{T}_{N_1}$ resp. $\mathcal{T}^*_{N_2}$ is a T-polynomial on E_{l_1} resp. $E^*_{l_2}$ with $a^*_1 = -1$ and $a^*_{2l_2} = 1$. Then the composition $\tilde{\mathcal{T}}^*_{N_2}(\tilde{\mathcal{T}}_{N_1})$ is a normed T-polynomial on a set of disjoint intervals $E^{**}_{l_3}$.*

Proof. (a) Necessity. From relation (2.1) it follows that

$$|\mathcal{T}_N(y_j)| = \max_{x \in E_l} |\mathcal{T}_N(x)|$$

at the $N+l$ zeros y_j of $H\mathcal{U}_{N-l}$. Assuming that there is an additional point $y^* \in E_l$, y^* no zero of $H\mathcal{U}_{N-l}$, such that $|\mathcal{T}_N(y^*)| = \max_{x \in E_l} |\mathcal{T}_N(x)|$ we get that $y^* \in \operatorname{int}(E_l)$ and thus $\mathcal{T}'_N(y^*) = 0$, which implies by Corollary 2.2(b) that $\mathcal{T}'_N$ has at least N zeros which is a contradiction.

Sufficiency. Since $\mathcal{T}'_N$ has at most $N-1$ zeros it follows that $|\mathcal{T}_N|$ attains its maximum at all boundary points of E_l. Hence

$$\prod_{j=1}^{N+l} (x - y_j) = H(x) \prod_{\mu=1}^{N-l} (x - y_{j_\mu})$$

and

$$\mathcal{T}^2_N(x) = H(x) \prod_{\mu=1}^{N-l} (x - y_{j_\mu})^2 + L^2,$$

where $L = \max_{x \in E_l} |\mathcal{T}_N(x)|$, which proves the sufficiency part.

(b) and (c) Follow immediately with the help of (a). ∎

COROLLARY 2.3. *Let $\mathcal{T}_N$ be a T-polynomial on E_l. Then*

(a) *$\mathcal{T}_N$ is the unique minimal polynomial on E_l with respect to the maximum norm; i.e., $\mathcal{T}_N$ deviates least from zero on E_l with respect to the* *maximum norm among all polynomials of degree N with leading coefficient one.*

(b) *Let $E_{l'}$ be a set of disjoint intervals with the properties that $E_{l'} \subset E_l$ and that $E_{l'}$ contains at least $N+1$ alternation points of $\mathscr{T}_N$, i.e., at least $N+1$ points $\tilde{y}_1 < \tilde{y}_2 < \cdots < \tilde{y}_{N+1}$, $\tilde{y}_j \in E_{l'}$, such that $\mathscr{T}_N(\tilde{y}_j) = (-1)^{N+1-j} \max_{x \in E_l} |\mathscr{T}_N(x)|$ for $j = 1, ..., N+1$. Then $\mathscr{T}_N$ is a minimal polynomial on $E_{l'}$ and there exists no T-polynomial of degree N on $E_{l'}$.*

Proof. (a) From Theorem 2.2(a) we deduce that there exist $N+1$ points $y_{j_\mu} \in E_l$, $y_{j_1} < y_{j_2} < \cdots < y_{j_{N+1}}$, such that

$$\mathscr{T}_N(y_{j_\mu}) = (-1)^{N+1-\mu} \max_{x \in E_l} |\mathscr{T}_N(x)| \qquad \text{for} \quad \mu = 1, ..., N+1.$$

The assertion follows now by the well known alternation and uniqueness theorem for compact sets.

(b) From the Alternation Theorem it follows that $\mathscr{T}_N$ is a minimal polynomial on $E_{l'}$. In view of the uniqueness of the minimal polynomial and in view of part (a), (b) follows. ∎

Remark 2.2. Since by the well known Alternation Theorem a polynomial of degree N with leading coefficient one is a minimal polynomial on E_l with respect to the maximum norm if and only if it has $N+1$ alternation points on E_l we conclude by Theorem 2.2(a) that a minimal polynomial need not be a T-polynomial on E_l. But it is not hard to demonstrate that each minimal polynomial on E_l is a T-polynomial on a set of l' disjoint intervals including E_l.

In what follows the next theorem is of great importance.

THEOREM 2.3. *Let $\mathscr{T}_N$ be a T-polynomial on E_l and assume that there is no T-polynomial on E_l of lower degree. Then the polynomials $\mathscr{T}_{nN}$, $n \in \mathbf{N}$, are the only T-polynomials on E_l.*

Proof. In view of Corollary 2.3(a) there are no other T-polynomials on E_l of degree nN, $n \in \mathbf{N}$.

Now assume that there is a T-polynomial $\mathscr{T}_m$, $(n-1)N < m < nN$, $n \geqslant 2$, on E_l. Then we obtain with the help of Lemma I.6, Section 5, that

$$(\tilde{\mathscr{T}}_m \tilde{\mathscr{T}}_{nN} - H\tilde{\mathscr{U}}_{m-l}\tilde{\mathscr{U}}_{nN-l})^2 - H(\tilde{\mathscr{T}}_m \tilde{\mathscr{U}}_{nN-l} - H\tilde{\mathscr{T}}_{nN}\tilde{\mathscr{U}}_{m-l})^2 = 1$$

and that $t_{nN-m} := \mathscr{T}_m \mathscr{T}_{nN} - H\mathscr{U}_{m-l}\mathscr{U}_{nN-l}$ is a polynomial of degree $nN - m < N$. Hence t_{nN-m} is a T-polynomial on E_l of degree less than N which is a contradiction. ∎

The next corollary shows the connection with certain elliptic resp. hyperelliptic integrals.

COROLLARY 2.4. *Let $\mathcal{T}_N$ be a T-polynomial on E_l with exactly (ν_j+1), $\nu_j \in \mathbf{N}$, extremal points in (a_{2j-1}, a_{2j}), $j=1, ..., l$, and let r_{l-1} be that polynomial defined in Corollary* 2.2(b). *Then*

$$\int_{a_{2j}}^{a_{2j+1}} \frac{r_{l-1}(x)}{\sqrt{|H(x)|}}\, dx = 0 \qquad \text{for} \quad j=1, ..., l-1$$

and

$$\int_{a_{2j-1}}^{a_{2j}} \frac{|r_{l-1}(x)|}{\sqrt{|H(x)|}}\, dx = \frac{\nu_j \pi}{N} \qquad \text{for} \quad j=1, ..., l.$$

Proof. Let $y(x) = \mathcal{T}_N(x)$. Then it follows from (2.1) and Corollary 2.2 that y satisfies the differential equation

$$\frac{r_{l-1}^2}{NH} = \frac{(y')^2}{y^2-1}. \tag{2.6}$$

Solving this differential equation on $[a_{2j}, a_{2j+1}]$, $j \in \{1, ..., l-1\}$, by using the facts that $|y| > 1$, $y(a_{2j}) = y(a_{2j+1}) = \pm 1$, and $\operatorname{sgn} y' = \operatorname{sgn} r_{l-1} \mathcal{U}_{N-l}$, we get that y is of the form

$$y(x) = \pm \cosh\left(N \int_{a_{2j}}^{x} \frac{r_{l-1}(t)}{\sqrt{|H(t)|}}\, dt \right) \qquad \text{for} \quad x \in [a_{2j}, a_{2j+1}].$$

Since $y(a_{2j}) = y(a_{2j+1}) = \pm 1$ the first relation follows.

Solving (2.6) for $x \in [t_1, t_2] \subset E_l$, where t_1, t_2 denotes two consecutive extremal values of y, we get that

$$\arccos(\pm y(x)) = N \int_{t_1}^{x} \frac{|r_{l-1}(t)|}{\sqrt{|H(t)|}}\, dt + 2k\pi,$$

$k \in \mathbf{N}$, from which it follows, since $y(t_1) = -y(t_2) = \mp 1$, that

$$\int_{t_1}^{t_2} \frac{|r_{l-1}(t)|}{\sqrt{|H(t)|}}\, dt = \frac{\pi}{N}$$

which gives the assertion. ∎

THEOREM 2.4. *Suppose that there exists a normed T-polynomial $\tilde{\mathcal{T}}_N \neq T_N$ on $\bigcup_{j=1}^{N} [a_{2j-1}, a_{2j}]$, where $-1 = a_1 < a_2 \leqslant a_3 < a_4 \leqslant a_5 < \cdots \leqslant a_{2N-1} < a_{2N} = 1$ and $\tilde{\mathcal{T}}'_N$ has no zero on $\bigcup_{j=1}^{N} (a_{2j-1}, a_{2j})$. Then each interval $[a_{2j-1}, a_{2j}]$, $j=2, ..., N-1$, contains at most one zero of U_{N-1} and the boundary intervals $[-1, a_2]$ and $[a_{2N-1}, 1]$ contain no zero of U_{N-1}.*

Proof. Let $\delta \in \{-1, 1\}$. Since, using the facts that $|T_N| \leqslant 1$ and that $\tilde{\mathscr{T}}_N(a_j) = (-1)^{2N-j}$ for $j = 1, ..., 2N$,

$$(-1)^{2N-j} \operatorname{sgn}(\tilde{\mathscr{T}}_N - \delta T_N)(a_j) \geqslant 0 \qquad \text{for} \quad j = 1, ..., 2N,$$

it follows immediately by Rolle's Theorem that $\tilde{\mathscr{T}}_N - \delta T_N$ has at least one zero in each interval $[a_{2j-1}, a_{2j}]$, $j = 1, ..., N$.

Now suppose to the contrary that there is an $i^* \in \{2, ..., N-1\}$ such that $I_{i^*} := [a_{2i^*-1}, a_{2i^*}]$ contains more than one zero of U_{N-1}. Let us recall that at the zeros $y_1 < y_2 < \cdots < y_{N-1}$ of $U_{N-1}\, T_N(y_j) = (-1)^{N-j}$ for $j = 1, ..., N$.

Case (1). I_{i^*} contains at least three zeros $y_\nu < y_{\nu+1} < y_{\nu+2}$, $\nu \in \{1, ..., N-3\}$, of U_{N-1}. Considering $\tilde{\mathscr{T}}_N - \delta T_N$ at these y_ν's and using the fact that $|\tilde{\mathscr{T}}_N| < 1$ on $\operatorname{int}(I_{i^*})$ we get again by Rolle's Theorem that $\tilde{\mathscr{T}}_N - \delta T_N$ has at least one zero in $[y_\nu, y_{\nu+1})$ and $(y_{\nu+1}, y_{\nu+2}]$.

Case (2). I_{i^*} contains exactly two zeros $y_\nu < y_{\nu+1}$, $\nu \in \{1, ..., N-2\}$, of U_{N-1}. Choosing δ such that

$$\operatorname{sgn} \tilde{\mathscr{T}}_N(a_{2i^*-1}) = \delta \operatorname{sgn} T_N(y_\nu)$$

we get, since

$$\operatorname{sgn}(\tilde{\mathscr{T}}_N - \delta T_N)(a_{2i^*-1}) \operatorname{sgn} \tilde{\mathscr{T}}_N(a_{2i^*-1}) \geqslant 0$$

and

$$-\delta \operatorname{sgn} T_N(y_\nu) \operatorname{sgn}(\tilde{\mathscr{T}}_N - \delta T_N)(y_\nu) > 0,$$

that $\tilde{\mathscr{T}}_N - \delta T_N$ has at least one zero in $[a_{2i^*-1}, y_\nu)$. Analogously one demonstrates that $\tilde{\mathscr{T}}_N - \delta T_N$ has at least one zero in $(y_\nu, a_{2i^*}]$.

In both cases there is a $\delta \in \{-1, +1\}$ such that $\tilde{\mathscr{T}}_N - \delta T_N$ has at least two zeros in $[a_{2i^*-1}, a_{2i^*}]$ and, as it was demonstrated above, at least one zero in each interval $[a_{2j-1}, a_{2j}]$, $j \in \{1, ..., N\} \setminus \{i^*\}$, and thus $N+1$ zeros which implies that $\tilde{\mathscr{T}}_N = T_N$ which is a contradiction. ∎

Next let us give another characterization of T-polynomials.

THEOREM 2.5. *The following propositions are equivalent*:

(a) *There exists a T-polynomial $\mathscr{T}_N$ on E_l.*

(b) *There are polynomials H^+, H^-, $\mathscr{U}^+$, $\mathscr{U}^-$ with leading coefficient one such that*

$$H^+(x)\, H^-(x) = H(x), \qquad \operatorname{sgn} H^\pm(x) = \pm 1 \text{ on } E_l \tag{2.7}$$

and

$$(\mathscr{T}_N =)\; H^+(\mathscr{U}^+)^2 - L = H^-(\mathscr{U}^-)^2 + L,$$

where $L \in \mathbf{R}^+$.

(c) *There are polynomials* $\mathscr{U}^+$, H^+, H^- *with leading coefficient one, such that* H^+, H^- *satisfy condition* (2.7) *and* $\mathscr{U}^+ \perp \mathbf{P}_{\partial\mathscr{U}^+ + l - 2}$ on E_l *with respect to* H^+/h.

(d) *There are polynomials* $\mathscr{U}^-$, H^+, H^- *with leading coefficient one, such that* H^+, H^- *satisfy condition* (2.7) *and* $\mathscr{U}^- \perp \mathbf{P}_{\partial\mathscr{U}^- + l - 2}$ *on* E_l *with respect to* H^-/h.

Proof. (a) $\Rightarrow$ (*b*). Let

$$H^{\pm}(x) = \prod (x - a_j^{\pm}) \qquad \text{and} \qquad \mathscr{U}^{\pm}(x) = \prod (x - y_j^{\pm}),$$

where $a_j^{\pm} \in \{a_1, ..., a_{2l}\}$ and $y_j^{\pm} \in \operatorname{int}(E_l)$ denotes that extremal points at which $\mathscr{T}_N$ attains its maximum value $\pm L$. Using the fact that by Theorem 2.2(a), $\mathscr{T}_N$ has $N + l$ extremal points the assertion follows.

(b) $\Rightarrow$ (c). In view of (b) we have

$$H^+(\mathscr{U}^+)^2 - H^-(\mathscr{U}^-)^2 = 2L. \tag{2.8}$$

By Theorem I.1 the implication is proved.

(c) $\Rightarrow$ (d). By the orthogonality property of $\mathscr{U}^+$ it follows from Theorem I.3 that there is a polynomial $\mathscr{U}^-$ with the given orthogonality property.

(d) $\Rightarrow$ (a). Again by Theorem I.3. it follows that there is a polynomial $\mathscr{U}^+$ such that (2.8) holds. Hence

$$\mathscr{T}_N = H^+(\mathscr{U}^+)^2 - L = H^-(\mathscr{U}^-)^2 + L$$

which implies that

$$\mathscr{T}_N^2 - H(\mathscr{U}^+\mathscr{U}^-)^2 = L^2. \quad \blacksquare$$

COROLLARY 2.5. *Let* H^+, H^- *be polynomials with leading coefficient one such that*

$$H^+H^- = H \qquad \text{and} \qquad \operatorname{sgn} H^{\pm} = \pm 1 \text{ on } E_l,$$

put

$$m_k^{\pm} = \int_{E_l} x^k H^{\pm}(x) \frac{dx}{h(x)} \qquad \text{for} \quad k \in \mathbf{N}_0,$$

and set for j, $\mu \in \mathbf{N}_0$

$$\det M_{j,\mu}^{\pm} = \begin{vmatrix} m_j^{\pm} & m_{j+1}^{\pm} & \cdots & m_{j+\mu}^{\pm} \\ m_{j+1}^{\pm} & m_{j+2}^{\pm} & \cdots & m_{j+\mu+1}^{\pm} \\ \vdots & \vdots & & \vdots \\ m_{j+\mu}^{\pm} & m_{j+\mu+1}^{\pm} & \cdots & m_{j+2\mu}^{\pm} \end{vmatrix}.$$

Then the following propositions are equivalent:

(a) *There exists a T-polynomial* $\mathscr{T}_N$ *on* E_l.

(b) $\det M^+_{j,\mu} = 0$ *for* $\mu = (N - \partial H^+)/2$ *and* $j = 0, 1, ..., l-2$.

(c) $\det M^-_{j,\kappa} = 0$ *for* $\kappa = (N - \partial H^-)/2$ *and* $j = 0, 1, ..., l-2$.

Proof. In view of Theorem 2.5 the assertion is equivalent to the following statement: There exists a polynomial $\mathscr{U}^{\pm} \perp \mathbf{P}_{\partial\mathscr{U}^{\pm}+l-2}$ on E_l with respect to $H^{\pm}/h$ if and only if the above given determinants are zero. This can be demonstrated analogously as in Theorem 2.1(b). ∎

Hence Corollary 2.5 gives a simpler condition for the existence of a T-polynomial than Theorem 2.1. For the calculation of the moments $m_k^{\pm}$ see (5.15) and (5.16).

3. Orthogonal Polynomials with Periodic Recurrence Coefficients

In the first part of this section we show that polynomials orthogonal with respect to $\Psi_{R,\rho,\varepsilon}$, $\Psi_{R,\rho,\varepsilon}$ definite, have periodic recurrence coefficients if there exists a T-polynomial $\mathscr{T}_N$ on E_l and give a representation of the orthogonal polynomials with the help of the T-polynomial $\mathscr{T}_N$. On the other hand we demonstrate that orthogonal polynomials having recurrence coefficients of period N are orthogonal on a union of $l \leqslant N$ disjoint intervals E_l with respect to a functional $\Psi_{R,\rho,\varepsilon}$, a result which has been given by Geronimus [7] using different methods.

We need the following

Notation. Let R, ρ_ν, ε be given and let $v \in \mathbf{P}_{\nu-1}$ and $u \in \mathbf{P}_{r-(\nu+l)}$ be such that at the zeros w_k of $\rho_\nu(x) = \prod_{k=1}^{\nu^*} (x - w_k)^{\nu_k}$

$$v^{(j)}(w_k) = \varepsilon_k (R/\sqrt{H})^{(j)}(w_k) \qquad \text{for} \quad j = 0, ..., \nu_k - 1,$$

and that

$$(R/\rho_\nu \sqrt{H})(z) = u(z) + 0(z^{-1})$$

and put

$$Y_{R,\rho_\nu,\varepsilon}(z) = u(z)\,\rho_\nu(z) + v(z), \tag{3.1}$$

where we shall omit the indices ν resp. R, ρ_ν, ε if there is no confusion possible.

Now assume that $\Psi_{R,\rho_\nu,\varepsilon}$ is definite (see, e.g., [3]); i.e., there exists a unique sequence of polynomials $(p_n = x^n + \cdots)_{n \in \mathbf{N}}$ such that

$\Psi_{R,\rho_v,\varepsilon}(x^j p_n)=0$ for $j=0, ..., n-1$ and $\Psi_{R,\rho_v,\varepsilon}(x^n p_n)\neq 0$. Then it is well known that (p_n) satisfies a recurrence relation of the form

$$p_n(x)=(x-\alpha_n)\,p_{n-1}(x)-\lambda_n p_{n-2}(x) \qquad \text{for} \quad n\in\mathbf{N}, \tag{3.2}$$

with

$$\alpha_n\in\mathbf{R} \qquad \text{and} \qquad \lambda_{n+1}\in\mathbf{R}\setminus\{0\} \qquad \text{for} \quad n\in\mathbf{N},$$

where $p_{-1}(x)=0$ and $p_0(x)=1$, and we denote as usual by $(p_n^{(j)})$ the associated polynomials of order j defined by

$$p_n^{(j)}(x)=(x-\alpha_{n+j})\,p_{n-1}^{(j)}(x)-\lambda_{n+j}\,p_{n-2}^{(j)}(x) \qquad \text{for} \quad n\in\mathbf{N}, \tag{3.3}$$

with $p_{-1}^{(j)}(x)=0$ and $p_0^{(j)}(x)=1$. Furthermore we put $m=n+r-l$ and

$$q_m(x)=Y(x)\,p_n(x)+\lambda_1\rho_v(x)\,p_{n-1}^{(1)}(x) \qquad \text{for} \quad n\in\mathbf{N}_0, \tag{3.4}$$

where

$$\lambda_1 := \Psi_{R,\rho_v,\varepsilon}(1). \tag{3.5}$$

Remark 3.1. (a) Let us recall that by Theorem I.3, $\partial q_m=n+r-l$ for $n \geqslant (v+l-r)/2$ and that $q_m \perp \mathbf{P}_{m-1}$ with respect to $\Psi_{S,\rho_v,\varepsilon}$ for $n\geqslant \max\{v, (v+l-r)/2\}$.

(b) Note that q_m satisfies the same recurrence relation with respect to n as p_n.

(c) Proving Theorem I.3 we have demonstrated that for $z\in\mathbf{C}\setminus E_l$, $|z|$ sufficiently large,

$$\frac{-Y_{R,\rho_v,\varepsilon}(z)+(R/\sqrt{H})(z)}{\rho_v(z)}=\Psi_{R,\rho_v,\varepsilon}\left(\frac{1}{z-x}\right). \tag{3.6}$$

The first main result of this section is

THEOREM 3.1. *Let $\mathscr{T}_N$ be a T-polynomial on E_l and let $\Psi_{R,\rho_v,\varepsilon}$ be definite. Suppose that (p_n) is orthogonal with respect to $\Psi_{R,\rho_v,\varepsilon}$ and satisfies a recurrence relation of the form* (3.2). *Furthermore put* $n_0 := \max\{0, v+1-N, [(v+l+1-r)/2]\}$.

Then the following propositions hold:

(a) *For $k\in\mathbf{N}$, $n\in\mathbf{N}_0$, $n\geqslant n_0$,*

$$2p_{kN+n}=p_n\mathscr{T}_{kN}+Sq_m\mathscr{U}_{kN-l},$$

and

$$2q_{kN+m}=q_m\mathscr{T}_{kN}+Rp_n\mathscr{U}_{kN-l},$$

where $\mathscr{T}_{kN}$ resp. $\mathscr{U}_{kN-l}$ are defined in (2.3) *resp.* (2.4).

(b) *The recurrence coefficients of* (p_n) *have period* N *for* $n \geqslant n_0$, *i.e., for* $n \geqslant n_0$

$$\alpha_{N+n+2} = \alpha_{n+2} \quad \text{and} \quad \lambda_{N+n+2} = \lambda_{n+2}.$$

(c) *If in addition* $v = r - l - 1 \geqslant 0$, *then*

$$p_{N-1} = \hat{\rho}_v S \mathscr{U}_{N-l} \quad \text{and} \quad 2p_N = \mathscr{T}_N + SY\mathscr{U}_{N-l},$$

and

$$\alpha_{N+1} = \alpha_1 \quad \text{and} \quad \lambda_{N+1} = K_{\rho_v} \lambda_1 / 2,$$

where K_{ρ_v} is the leading coefficient of ρ_v.

Proof. (a) Since by Theorem I.3 (see Lemma I.6 and Theorem I.4 for details)

$$\begin{aligned} R(p_n \mathscr{T}_{kN} + Sq_m \mathscr{U}_{kN-l})^2 - S(q_m \mathscr{T}_{kN} + Rp_n \mathscr{U}_{kN-l})^2 \\ = (\mathscr{T}^2_{kN} - H\mathscr{U}^2_{kN-l})(Rp_n^2 - Sq_m^2) = L_{kN} \rho g_{(n)}, \end{aligned}$$

where $g_{(n)} \in \mathbf{P}_{l-1}$ and at the zeros w_k of ρ

$$(R(p_n \mathscr{T}_{kN} + Sq_m \mathscr{U}_{kN-l}))(w_k) = \varepsilon_k \sqrt{H(w_k)} (q_m \mathscr{T}_{kN} + Rp_n \mathscr{U}_{kN-l})(w_k).$$

part (a) follows from Theorem I.1.

(b) Observing that q_{m+2} satisfies the same recurrence relation as p_{n+2} for $n \geqslant n_0$ we get that for $n \geqslant n_0$

$$\begin{aligned} p_{n+2} \mathscr{T}_N + Sq_{m+2} \mathscr{U}_{N-l} \\ = (x - \alpha_{n+2})(p_{n+1} \mathscr{T}_N + Sq_{m+1} \mathscr{U}_{N-l}) - \lambda_{n+2}(p_n \mathscr{T}_N + Sq_m \mathscr{U}_{N-l}), \end{aligned}$$

which gives in conjunction with (a) that

$$p_{N+n+2} = (x - \alpha_{n+2}) p_{N+n+1} - \lambda_{n+2} p_{N+n}$$

from which the assertion follows.

(c) First let us note that $n_0 = 0$ and by (3.4), $q_{r-l} = Y$. Since on the one hand

$$p_1 \mathscr{T}_N + Sq_{r+1-l} \mathscr{U}_{N-l} = (x - \alpha_1)(\mathscr{T}_N + SY\mathscr{U}_{N-l}) + \lambda_1 S \rho \mathscr{U}_{N-l}$$

and on the other hand

$$p_{N+1} = (x - \alpha_{N+1}) p_N - \lambda_{N+1} p_{N-1}$$

the assertion follows with the help of (a). ∎

Next we need some general facts on orthogonal polynomials.

LEMMA 3.1. *Suppose that $(p_n)_{n\in N_0}$ satisfies a recurrence relation of the form* (3.2). *Then the following propositions hold for $k, n\in \mathbf{N}_0$:*

(a) $p_n^{(k)}=(x-\alpha_{k+1})\,p_{n-1}^{(k+1)}-\lambda_{k+2}\,p_{n-2}^{(k+2)}$.

(b) *For $j\in \mathbf{N}_0$*

$$p_{j-1}^{(n+1+k)}p_{n+j-1}^{(k)}-p_{j-2}^{(n+1+k)}p_{n+j}^{(k)}=\left(\prod_{\mu=n+2}^{n+j}\lambda_{\mu+k}\right)p_n^{(k)}$$

which is the so-called Wronskian formula.

(c) *For $j\in\{1,\ldots,n-1\}$*

$$p_n^{(k)}=p_j^{(n+k-j)}p_{n-j}^{(k)}-\lambda_{n+k+1-j}\,p_{j-1}^{(n+k+1-j)}p_{n-j-1}^{(k)}$$

and this representation is unique for $2j\leqslant n$; i.e., if $2j\leqslant n$ and

$$p_n^{(k)}=u_j p_{n-j}^{(k)}-v_{j-1}p_{n-j-1}^{(k)},$$

where $u_j\in \mathbf{P}_j$, $v_{j-1}\in \mathbf{P}_{j-1}$, then

$$u_j=p_j^{(n+k-j)}\qquad \textit{and}\qquad v_{j-1}=\lambda_{n+k+1-j}p_{j-1}^{(n+k+1-j)}.$$

Proof. (a) This has been given in [4].

(b) This is known and can be demonstrated by induction using the recurrence relation of $p_j^{(n+1)}$ and p_{n+j+1}.

(c) The first assertion follows by induction arguments again using the recurrence relation of $p_{n-j}^{(k)}$ and (a).

Concerning the uniqueness of the representation we get that at the $n-j\geqslant j$ zeros of $p_{n-j}^{(k)}$

$$v_{j-1}=p_n^{(k)}/p_{n-j-1}^{(k)}$$

which gives in conjunction with the first representation of $p_n^{(k)}$ the assertion. ∎

LEMMA 3.2. *Suppose that (p_n) satisfies a recurrence relation of the form* (3.2). *Let k, $N\in\mathbf{N}$, $n\in\mathbf{N}_0$, and assume that*

$$p_{(k+1)N+n}(x)=a(x)\,p_{kN+n}(x)-cp_{(k-1)N+n}(x),\tag{3.7}$$

where $a\in\mathbf{P}_N$ and $c\in\mathbf{R}$. Then the following proposition holds:

$$a=p_N^{(kN+n)}-\lambda_{kN+n+1}\,p_{N-2}^{((k-1)N+n+1)},$$

$$c=\prod_{j=(k-1)N+n+2}^{kN+n+1}\lambda_j,$$

$$p_{N-1}^{(kN+n+1)}=p_{N-1}^{((k-1)N+n+1)}.$$

Proof. Put

$$K = \prod_{j=(k-1)N+n+2}^{kN+n} \lambda_j.$$

Then it follows from Lemma 3.1(b) that

$$Kp_{(k-1)N+n} = p_{N-1}^{((k-1)N+n+1)} p_{kN+n-1} - p_{N-2}^{((k-1)N+n+1)} p_{kN+n}.$$

Thus we get from (3.7) that

$$p_{(k+1)N+n} = \left(a + \frac{c}{K} p_{N-2}^{((k-1)N+n+1)}\right) p_{kN+n}$$

$$- \frac{c}{K} p_{N-1}^{((k-1)N+n+1)} p_{kN+n-1}.$$

Since on the other hand by Lemma 3.1(c)

$$p_{(k+1)N+n} = p_N^{(kN+n)} p_{kN+n} - \lambda_{kN+n+1} p_{N-1}^{(kN+n+1)} p_{kN+n-1}$$

the assertion follows by the uniqueness of the representation. ∎

COROLLARY 3.1. *Suppose that the assumptions of Theorem* 3.1 *are fulfilled. Then the following propositions hold:*

(a) *For each* $k \in \mathbf{N}_0$ *and each* $n \geqslant n_0$

$$p_{(k+2)N+n} = \mathcal{T}_N p_{(k+1)N+n} - (L^2/4)\, p_{kN+n}$$

and

$$q_{(k+2)N+m} = \mathcal{T}_N q_{(k+1)N+m} - (L^2/4)\, q_{kN+m}.$$

(b) *For each* $n \geqslant n_0 + 1$ *resp.* $n \geqslant n_0$

$$p_N^{(n)} - \lambda_{N+n+1} p_{N-2}^{(n+1)} = \mathcal{T}_N \qquad \textit{resp.} \quad 4 \prod_{j=n+2}^{N+n+1} \lambda_j = L^2,$$

where the first relation holds also for $n = n_0$ *if* $\alpha_{N+n_0+1} = \alpha_{n_0+1}$.

(c) *For each* $n \geqslant n_0$

$$\prod_{j=1}^{n+1} \lambda_j \rho_v\, S\mathcal{U}_{N-l} = 2(p_{N+n+1} p_n - p_{N+n} p_{n+1}).$$

(d) *For each* $n \geqslant n_0$

$$\prod_{j=2}^{n+1} \lambda_j (S\mathcal{U}_{N-l} Y + \mathcal{T}_N) = 2(p_n^{(1)} p_{N+n} - p_{n-1}^{(1)} p_{N+n+1}).$$

Proof. (a) For $k \in \mathbf{N}$ the assertion follows immediately from the recurrence relations

$$\mathcal{T}_{(k+2)N} = \mathcal{T}_N \mathcal{T}_{(k+1)N} - (L^2/4)\, \mathcal{T}_{kN} \qquad \text{for all} \quad k \in \mathbf{N}, \tag{3.8}$$

and

$$\mathcal{U}_{(k+2)N} = \mathcal{T}_N \mathcal{U}_{(k+1)N} - (L^2/4)\, \mathcal{U}_{kN} \qquad \text{for all} \quad k \in \mathbf{N}, \tag{3.9}$$

and Theorem 3.1(a).

Using the fact that

$$\mathcal{T}_{2N} = \mathcal{T}_N^2 - L^2/2$$

we obtain from Theorem 3.1(a) that

$$2p_{2N+n} = \mathcal{T}_N(p_n \mathcal{T}_N + Sq_m \mathcal{U}_{N-l}) - (L^2/2)\, p_n$$

which proves the assertion for $k = 0$.

(b) From part (a) and Lemma 3.2 it follows that

$$p_N^{(N+n)} - \lambda_{N+n+1}\, p_{N-2}^{(n+1)} = \mathcal{T}_N \qquad \text{for} \quad n \geqslant n_0,$$

and that the second relation holds. Since by the periodicity of the recursion coefficients

$$p_j^{(kN+n)} = p_j^{(n)} \qquad \text{for} \quad j, k \in \mathbf{N}_0 \text{ and } n \geqslant n_0 + 1$$

part (b) is proved.

(c) With the help of (3.4) we get by simple calculation that

$$\begin{aligned} p_n q_{m+1} - p_{n+1} q_m &= \lambda_1 \rho_\nu (p_n p_n^{(1)} - p_{n-1}^{(1)} p_{n+1}) \\ &= \prod_{j=1}^{n+1} \lambda_j \rho_\nu. \end{aligned}$$

Hence using the representation of $S\mathcal{U}_{N-l} q_j$, $j \in \{m, m+1\}$, from Theorem 3.1(a) we obtain

$$\prod_{j=1}^{n+1} \lambda_j \rho_\nu S\mathcal{U}_{N-l} = p_n(2p_{N+n+1} - p_{n+1} \mathcal{T}_N) - p_{n+1}(2p_{N+n} - p_n \mathcal{T}_N)$$

which is the assertion.

(d) Using again the representation of $S\mathcal{U}_{N-l} q_j$, $j = m$, $m+1$, from Theorem 3.1(a) we get that

$$\begin{aligned} &S\mathcal{U}_{N-l}(p_n^{(1)} q_m - p_{n-1}^{(1)} q_{m+1}) \\ &\quad = 2(p_n^{(1)} p_{N+n} - p_{N+n+1} p_{n-1}^{(1)}) - \prod_{j=2}^{n+1} \lambda_j \mathcal{T}_N \end{aligned}$$

which gives with the help of (3.4) the assertion. ∎

Corollary 3.1 could have also been derived directly from [5, Lemma 1, Corollary 1, Lemma 10, and Theorem 5] in which it has been shown that orthogonal polynomials with periodic recurrence coefficients satisfy the relations given in Corollary 3.1 in which $\mathcal{T}_N$ is to replace by its representation in terms of orthogonal polynomials. As we have learned Lemma 3.2 could also be obtained from Lemma 11 and Eq. (V.4) in [5].

COROLLARY 3.2. *Suppose that the assumptions of Theorem* 3.1 *are fulfilled. Then the following propositions hold for each* $n \geqslant n_0 + 1$:

(a) $p_{N-1}^{(n)} = \mathcal{U}_{N-l}\,\hat{g}_{(n-1)}$, *where* $\hat{g}_{(n-1)}$ *has no zero in* $\operatorname{int}(E_l)$.

(b) $p_N^{(n)} + \lambda_{N+n+1}\, p_{N-2}^{(n+1)} = \mathcal{U}_{N-l}\,\hat{f}_{(n)}$.

Proof. (a) Since on one hand, using the representation of q_{N+m-1} and q_{m-1} from Theorem 3.1 and using Theorem I.3

$$q_{N+m-1}p_{n-1} - p_{N+n-1}q_{m-1} = \mathcal{U}_{N-l}(Rp_{n-1}^2 - Sq_{m-1}^2) = \mathcal{U}_{N-l}\,\rho_\nu\, g_{(n-1)} \tag{3.10}$$

and on the other hand by simple calculation

$$\begin{aligned} q_{N+m-1}p_{n-1} - p_{N+n-1}q_{m-1} &= \lambda_1 \rho_\nu (p_{N+n-2}^{(1)} p_{n-1} - p_{N+n-1} p_{n-2}^{(1)}) \\ &= \prod_{j=1}^{n} \lambda_j \rho_\nu p_{N-1}^{(n)}, \end{aligned} \tag{3.11}$$

where the last equality follows from relation (10) of [4] the first assertion of part (a) is proved.

Concerning the second assertion let us assume to the contrary that $g_{(n-1)}$ has a zero y in $\operatorname{int}(E_l)$. Since $-H>0$ on $\operatorname{int}(E_l)$ it follows from the second relation of (3.10) that $p_n(y) = q_m(y) = 0$ which implies by the definition of q_m, taking into account the fact that p_n and $p_{n-1}^{(1)}$ have no common zero, that $\rho_\nu(y) = 0$ which is a contradiction.

(b) With the help of the first relation of Corollary 3.1(b) we get

$$[p_N^{(n)} + \lambda_{N+n+1}\, p_{N-2}^{(n+1)}]^2 - H\mathcal{U}_{N-l}^2 = L^2 + 4\lambda_{N+n+1}\, p_N^{(n)} p_{N-2}^{(n+1)}$$

which gives in conjunction with the second relation of Corollary 3.1(b) and Lemma 3.1(b) that for $n \geqslant n_0 + 1$

$$[p_N^{(n)} + \lambda_{N+n+1}\, p_{N-2}^{(n+1)}]^2 - H\mathcal{U}_{N-l}^2 = 4\lambda_{N+n+1}\, p_{N-1}^{(n)}\, p_{N-1}^{(n+1)}$$

which proves, in view of (a), part (b). ∎

In Section 5 we shall demonstrate how to obtain from the relations of Corollary 3.2 a nonlinear recurrence relation for the recursion coefficients of p_n, if $N > l$.

THEOREM 3.2. *Suppose that the assumptions of Theorem* 3.1 *are fulfilled and let* $j \geq n_0 + 1$. *Let* $w_k^{(j)}$, $k = 1, ..., v^{(j)}$, be the zeros of $p_{N-1}^{(j)}/\mathscr{U}_{N-l}$ *and put*

$$\varepsilon_k^{(j)} = \operatorname{sgn}((p_N^{(j)} + \lambda_{N+j+1} p_{N-2}^{(j+1)})/\mathscr{U}_{N-l}\sqrt{H})(w_k^{(j)})$$

for $k = 1, ..., v^{(j)}$. *Then*

(a) *The associated polynomials* $(p_n^{(j)})_{n \in N_0}$ *are orthogonal with respect to* $\Psi_{H, p_{N-1}^{(j)}/\mathscr{U}_{N-l}, \varepsilon^{(j)}}$.

(b) *The polynomials* $((2p_{N+n}^{(j)} - \mathscr{T}_N p_n^{(j)})/\mathscr{U}_{N-l})_{n \in N_0}$ *are orthogonal on* E_l *with respect to* $\Psi_{p_{N-1}^{(j)}/\mathscr{U}_{N-l}, \varepsilon^{(j)}}$.

Proof. In view of (3.10) and (3.11) we have, putting $i = j + r - l$,

$$Rp_{j-1}^2 - Sq_{i-1}^2 = \rho_v g_{(j-1)} = \rho_v p_{N-1}^{(j)}/\mathscr{U}_{N-l},$$

where at the zeros $w_k^{(j)}$ of $p_{N-1}^{(j)}/\mathscr{U}_{N-l}$

$$(Rp_{j-1})(w_k^{(j)}) = -\delta_k^{(j)}(\sqrt{H}\, q_{i-1})(w_k^{(j)}),$$

$\delta_k^{(j)} \in \{-1, +1\}$. Thus, by Theorem I.5, it remains only to demonstrate that $-\delta_k^{(j)} = \varepsilon_k^{(j)}$. Since by Theorem I.5(c) and Theorem I.3

$$q_m^{(j)}(w_k^{(j)}) = \delta_k^{(j)}(\sqrt{H}\, p_n^{(j)})(w_k^{(j)}) \qquad \text{for all} \quad n \in \mathbf{N}_0, \tag{3.12}$$

where $m = n + l$, and since by the first relation of Theorem 3.1(a)

$$\mathscr{U}_{N-l} q_m^{(j)} = 2p_{N+n}^{(j)} - \mathscr{T}_N p_n^{(j)} \qquad \text{for} \quad n \in \mathbf{N}_0$$

and moreover, using Corollary 3.1(b),

$$\mathscr{U}_{N-l} q_l^{(j)} = p_N^{(j)} + \lambda_{N+j+1} p_{N-2}^{(j+1)}$$

the assertion follows by (3.12). ∎

Remark 3.2. Corollary 3.2 and Theorem 3.2 hold also for $n = n_0$ if $\alpha_{N+n_0+1} = \alpha_{n_0+1}$.

Remark 3.3. Theorem 3.1 a.s.o. could also be extended to the most general case where $\Psi_{R,\rho,\varepsilon}$ is not necessarily definite using the fact (see [16]) that for given R, ρ, ε there is a unique sequence of polynomials $p_{i_n}=x^{i_n}+\cdots$ with $\Psi_{R,\rho,\varepsilon}(x^j p_{i_n})=0$ for $j=0,...,i_{n+1}-2$ and $\Psi_{R,p,\varepsilon}(x^{i_{n+1}-1}p_{i_n})\neq 0$ satisfying a recurrence relation of the form

$$p_{i_n}=d_{i_n}p_{i_{n-1}}-\lambda_{i_n}p_{i_{n-2}},$$

where $d_{i_n}\in \mathbf{P}_{i_n-i_{n-1}}$ and $\lambda_{i_n}\in \mathbf{R}\setminus\{0\}$.

Next let us demonstrate that the converse of Theorem 3.1(b) holds also.

LEMMA 3.3. *Let $\alpha_n\in\mathbf{R}$, $\lambda_{n+1}\in\mathbf{R}\setminus\{0\}$, for $n\in\mathbf{N}$ and suppose that*

$$\alpha_{N+n+1}=\alpha_{n+1} \qquad \text{and} \qquad \lambda_{N+n+1}=\lambda_{n+1} \qquad \text{for} \quad n\geqslant n_0+1, \tag{3.13}$$

where N, $n_0\in\mathbf{N}$. Let (p_n) be the polynomials generated by the recurrence coefficients (α_n) and (λ_{n+1}) and put

$$\mathscr{T}_N:=p_N^{(n_0+1)}-\lambda_{N+n_0+2}\,p_{N-2}^{(n_0+2)} \qquad \text{and} \qquad L^2:=4\prod_{j=n_0+2}^{N+n_0+1}\lambda_j.$$

Furthermore let us assume that $L^2>0$, that $\mathscr{T}_N$ can be represented in the form

$$\mathscr{T}_N^2(x)=H(x)\,\mathscr{U}_{N-l}^2(x)+L^2, \tag{3.14}$$

where $H(x)=\prod_{j=1}^{2l}(x-a_j)$ with $a_1<a_2<\cdots<a_{2l}$, $\mathscr{U}_{N-l}\in\mathbf{P}_{N-l}$, and that $p_{N-1}^{(n_0+1)}$ has simple zeros at the zeros of $\mathscr{U}_{N-l}$. Then the following propositions hold:

(a) *$\mathscr{T}_N$ is a T-polynomial on $E_l:=\bigcup_{j=1}^{l}[a_{2j-1},a_{2j}]$ and $\mathscr{U}_{N-l}$ has $N-l$ simple zeros in* $\operatorname{int}(E_l)$.

(b) *$\mathscr{T}_N=p_N^{(n)}-\lambda_{N+n+1}\,p_{N-2}^{(n+1)}$ for each $n\geqslant n_0+1$.*

(c) *For each $n\geqslant n_0+1$*

$$p_{N-1}^{(n)}=\mathscr{U}_{N-l}\,\hat{g}_{(n-1)},$$

where $\hat{g}_{(n-1)}$ has no zero in $\operatorname{int}(E_l)$.

(d) *For each $n\geqslant n_0+2$*

$$p_n p_{N+n-1}-p_{n-1}p_{N+n}=\left(\prod_{j=n_0+2}^{n}\lambda_j\right)(p_{n_0+1}p_{N+n_0}-p_{n_0}p_{N+n_0+1})$$

and $p_{n_0+1}p_{N+n_0}-p_{n_0}p_{N+n_0+1}$ vanishes at the zeros of $\mathscr{U}_{N-l}$.

(e) *If in addition $\alpha_{N+n_0+1}=\alpha_{n_0+1}$, then* (b) *and* (c) *hold also for $n=n_0$.*

Proof. (a) Follows immediately from (2.1) and Corollary 2.2(a).

(b) See, e.g., [5, Lemma 1]. The assertion could also be proved with the help of Lemma 3.1(c) and (b) using relation (3.13).

(c) With the help of (b), relation (3.14), and Lemma 3.2(b) we get that for $n \geqslant n_0 + 1$

$$\begin{aligned}(p_N^{(n)} + \lambda_{N+n+1} p_{N-2}^{(n+1)})^2 - H\mathscr{U}_{N-l}^2 &\\ = \mathscr{T}_N^2 - H\mathscr{U}_{N-l}^2 + 4\lambda_{N+n+1} p_N^{(n)} p_{N-2}^{(n+1)} &\\ = 4\lambda_{N+n+1} p_{N-1}^{(n)} p_{N-1}^{(n+1)}. & \qquad (3.15)\end{aligned}$$

In view of (a), $\mathscr{U}_{N-l}$ has $N-l$ simple zeros in $\operatorname{int}(E_l)$. Thus we get from (3.15) by induction arguments that $p_{N-1}^{(n)}$ has simple zeros at the zeros of $\mathscr{U}_{N-l}$ for each $n \geqslant n_0+1$. Since $-H>0$ on $\operatorname{int}(E_l)$ we obtain from (3.15) that the zeros of $\mathscr{U}_{N-l}$ are the only zeros of $p_{N-1}^{(n)}$, $n \geqslant n_0+1$, lying in $\operatorname{int}(E_l)$, which proves (c).

(d) The first relation follows by induction.

Concerning the second assertion it follows with the help of the relations (see Lemma 3.1(c))

$$p_{N+n+j} = p_{N-1+j}^{(n+1)} p_{n+1} - \lambda_{n+2} p_{N-2+j}^{(n+2)} p_n$$

for $j=0, 1$, that at the zeros of $\mathscr{U}_{N-l}$, which are by (c) the common zeros of $p_{N-1}^{(n+1)}$ and $p_{N-1}^{(n+2)}$,

$$p_{n+1} p_{N+n} - p_n p_{N+n+1} = -p_n p_{n+1} (p_N^{(n+1)} + \lambda_{n+2} p_{N-2}^{(n+2)}).$$

Since by (3.15) and (c) the last expression vanishes at the zeros of $\mathscr{U}_{N-l}$ part (d) is proved.

(e) Since, by assumption,

$$\begin{aligned}p_N^{(n_0+1)} - \lambda_{N+n_0+2} p_{N-2}^{(n_0+2)} &\\ = (x - \alpha_{n_0+1}) p_{N-1}^{(n_0+1)} - \lambda_{N+n_0+1} p_{N-2}^{(n_0+1)} - \lambda_{n_0+2} p_{N-2}^{(n_0+2)},\end{aligned}$$

(b) holds also for $n = n_0$. Observing that (3.15) holds also for $n = n_0$, (c) follows. ∎

Remark 3.4. If the recurrence coefficients (α_n), (λ_{n+1}) satisfy the relations given in (3.13) and if $\lambda_{n+1} > 0$ for each $n \geqslant n_0+1$, then $\mathscr{T}_N := p_N^{(n_0+1)} - \lambda_{N+n_0+2} p_{N-2}^{(n_0+2)}$ satisfies relation (3.14) and $p_{N-1}^{(n)} = \mathscr{U}_{N-l} \hat{g}_{(n-1)}$ for $n \geqslant n_0+1$, where $\hat{g}_{(n-1)}$ has exactly one zero in each interval $[a_{2j}, a_{2j+1}]$, $j=1, ..., l-1$. This fact is due to Geronimus [7] (see [5, Lemma 2]).

THEOREM 3.3. *Suppose that the assumptions of Lemma 3.3 are fulfilled and let $\mathscr{T}_N$, $\mathscr{U}_{N-l}$, H, and E_l be defined as in Lemma 3.3. Furthermore put*

$$\rho(x)=(p_{n_0+1}p_{N+n_0}-p_{n_0}p_{N+n_0+1})/\mathscr{U}_{N-l}. \tag{3.16}$$

and set

$$\varepsilon_k=\operatorname{sgn}\left[\left(\frac{2p_{N+n_0}}{p_{n_0}}-\mathscr{T}_N\right)\Big/\mathscr{U}_{N-l}\sqrt{H}\right](w_k) \qquad \text{for} \quad k=1,...,\nu^*, \tag{3.17}$$

where $w_1, ..., w_{\nu^*}$ are the zeros of ρ. Then

(a) *The polynomials $(p_n)_{n\in\mathbf{N}}$ generated by the given periodic recurrence coefficients (α_n) and (λ_{n+1}) are orthogonal to $\mathbf{P}_{n-1}$ on E_l with respect to $\Psi_{H,\rho,\varepsilon}$.*

(b) *The polynomials $((2p_{N+n}-\mathscr{T}_N p_n)/\mathscr{U}_{N-l})_{n\in\mathbf{N}}$, are orthogonal to $\mathbf{P}_{n+l-1}$ on E_l with respect to $\Psi_{\rho,\varepsilon}$.*

Proof. ad (a) and (b). By (3.14) and Lemma 3.3(b) it follows using the relation (see Lemma 3.1(c))

$$p_{N+n}=p_N^{(n)}p_n-\lambda_{n+1}p_{N-1}^{(n+1)}p_{n-1}$$

that for $n\geqslant n_0+1$

$$\begin{aligned}(2p_{N+n}&-\mathscr{T}_N p_n)^2-H(\mathscr{U}_{N-l}p_n)^2\\ &=-4\lambda_{N+n+1}p_{N-1}^{(n+1)}p_{n-1}p_{N+n}+4\lambda_{N+n+1}p_n(p_{N-2}^{(n+1)}p_{N+n}+L^2p_n).\end{aligned}$$

Since by Lemma 3.1(b)

$$p_{N-1}^{(n+1)}p_{n+N-1}-p_{N-2}^{(n+1)}p_{N+n}=\left(\prod_{j=n+2}^{N+n}\lambda_j\right)p_n$$

we get with the help of the periodicity of the λ_j's, Lemma 3.3(d) and the definition of ρ that for $n\geqslant n_0+1$

$$[(2p_{N+n}-\mathscr{T}_N p_n)/\mathscr{U}_{N-l}]^2-Hp_n^2=\left(4\prod_{j=n_0+2}^{n+1}\lambda_j\right)p_{N-1}^{(n+1)}\rho/\mathscr{U}_{N-l}. \tag{3.18}$$

From (3.18) we get by simple calculation that at the zeros w_k of ρ

$$\left(\frac{2p_{N+n}}{p_n}-\mathscr{T}_N\right)(w_k)=\varepsilon_k^{(n)}\sqrt{H(w_k)}\,\mathscr{U}_{N-l}(w_k), \tag{3.19}$$

where $\varepsilon_k^{(n)}\in\{-1,+1\}$. Since by Lemma 3.3(d) at the zeros w_k of ρ

$$\left(\frac{p_{N+n_0}}{p_{n_0}}\right)(w_k)=\left(\frac{p_{N+n}}{p_n}\right)(w_k) \qquad \text{for} \quad n\geqslant n_0,$$

we obtain that

$$\varepsilon_k^{(n)} = \varepsilon_k^{(n_0)} \qquad \text{for} \quad n \geqslant n_0. \tag{3.20}$$

Applying Theorem I.1 to (3.18) and (3.19) the assertion is proved. ▮

Remark 3.5. Suppose that the assumptions of Theorem 3.3 are fulfilled and let ε_k be defined as in (3.17). Then, $k \in \{1, ..., v^*\}$,

$$\varepsilon_k = +1 \qquad \text{if and only if } \left|\frac{2p_{N+n_0}(w_k)}{p_{n_0}(w_k)}\right| < L \tag{3.21}$$

and

$$\varepsilon_k = -1 \qquad \text{if and only if } \left|\frac{2p_{N+n_0}(w_k)}{p_{n_0}(w_k)}\right| > L.$$

Proof. From Corollary 2.2(a) and (b) we obtain that

$$\operatorname{sgn} \mathscr{T}_N = \operatorname{sgn} \mathscr{U}_{N-l}\sqrt{H} \qquad \text{on} \quad \mathbf{R}\backslash E_l,$$

and hence by (2.1)

$$|\mathscr{T}_N + \sqrt{H}\mathscr{U}_{N-l}| > L \qquad \text{on} \quad \mathbf{R}\backslash E_l$$

and

$$|\mathscr{T}_N - \sqrt{H}\mathscr{U}_{N-l}| < L \qquad \text{on } \mathbf{R}\backslash E_l.$$

Since, by (3.19),

$$\frac{2p_{N+n_0}(w_k)}{p_{n_0}(w_k)} = (\mathscr{T}_N + \varepsilon_k\sqrt{H}\mathscr{U}_{N-l})(w_k)$$

the assertion is proved. ▮

Using different methods Theorem 3.3(a) has been given by Geronimus [7] (see also [5]) for the (probably most important) case that $\lambda_{n+1} \in \mathbf{R}^+$ for $n \in \mathbf{N}$. Instead of condition (3.17) Geronimus has given condition (3.21). Note that $\lambda_{n+1} \in \mathbf{R}^+$ for $n \in \mathbf{N}$ implies by Favard's Theorem that $\Psi_{H,\rho,\varepsilon}$ is positive definite and thus is of the form

$$\Psi_{H,\rho_v,\varepsilon}(p) = \int_{E_l} p(x)\frac{\sqrt{-H(x)}}{\pi\,|\rho_v(x)|}\,dx + \sum_{k=1}^{v^*} \mu_k\, p(w_k),$$

where $\operatorname{sgn} \rho_v = -\operatorname{sgn} h$ on $\operatorname{int}(E_l)$ and either $\mu_k = 0$ or $\mu_k = 2\sqrt{H(w_k)}/p_v'(w_k) > 0$.

THEOREM 3.4. *Let $\mathcal{T}_N$ be a T-polynomial on E_l and let $\alpha_n = \alpha_{N+n}$ and $\lambda_{n+1} = \lambda_{N+n+1} > 0$ for $n \in \mathbf{N}$ be the periodic recurrence coefficients of those polynomials which are orthogonal on E_l with respect to $\Psi_{H, p_{N-1}/\mathcal{U}_{N-l}, \varepsilon}$. Furthermore let us put for arbitrary $\tilde{\lambda}_N \in (0, \lambda_{N+1} + \lambda_N) \backslash \{\lambda_N\}$*

$$\tilde{\lambda}_{N+1} = \lambda_{N+1} + \lambda_N - \tilde{\lambda}_N \qquad \textit{and} \qquad L^2(\tilde{\lambda}_N) = \tilde{\lambda}_{N+1} \tilde{\lambda}_N L^2 / \lambda_{N+1} \lambda_N.$$

Then the following propositions hold:

(a) *The polynomials $(\widetilde{p_n})$ generated by the periodic recurrence coefficients $\alpha_1, ..., \alpha_N$, $\lambda_2, ..., \lambda_{N-1}, \tilde{\lambda}_N, \tilde{\lambda}_{N+1}$, are orthogonal on the N disjoint intervals $E(L(\tilde{\lambda}_N)) := \{x \in \mathbf{R} : |\mathcal{T}_N(x)| \leqslant L(\tilde{\lambda}_N)\} = \bigcup_{j=1}^{N} [a_{2j-1}(\tilde{\lambda}_N), a_{2j}(\tilde{\lambda}_N)]$ with respect to $\Psi_{\tilde{H}, p_{N-1}, \tilde{\varepsilon}}$, where $\tilde{H}(x) = \prod_{j=1}^{2N} (x - a_j(\tilde{\lambda}_N))$ and*

$$\tilde{\varepsilon}_k = \operatorname{sgn}\left(\frac{p_{N-2}(w_k)}{p_{N-2}^{(1)}(w_k)} - \frac{\tilde{\lambda}_{N+1}}{\tilde{\lambda}_N}\right) \qquad \textit{for} \quad k = 1, ..., N-1 \tag{3.22}$$

where w_k denotes the $N-1$ simple zeros of p_{N-1}.

(b) *The polynomials $(2\tilde{p}_{N+n} - \mathcal{T}_N \tilde{p}_n)_{n \in \mathbf{N}}$ are orthogonal to $\mathbf{P}_{N+n-1}$ on $E(L(\tilde{\lambda}_N))$ with respect to $\Psi_{p_{N-1}, \tilde{\varepsilon}}$, where $\tilde{\varepsilon}$ is defined in* (3.22).

(c) *If $\tilde{\lambda}_N > (<) \lambda_N$ then $\Psi_{\tilde{H}, p_{N-1}, \tilde{\varepsilon}}$ and $\Psi_{p_{N-1}, \tilde{\varepsilon}}$ have no (have a) point measure at all zeros of $\mathcal{U}_{N-l}$ and have no (have a) point measure at a zero w_j of $p_{N-1}/\mathcal{U}_{N-l}$ if $\Psi_{H, p_{N-1}/\mathcal{U}_{N-l}, \varepsilon}$ has no (has a) point measure at w_j.*

Proof. (a) and (b). In view of the assumption

$$\tilde{p}_{N-1-j}^{(j)} = p_{N-1-j}^{(j)} \text{ for } j = 0, ..., N-1 \qquad \text{and} \qquad \tilde{\lambda}_N + \tilde{\lambda}_{N+1} = \lambda_N + \lambda_{N+1} \tag{3.23}$$

which implies that

$$\tilde{p}_N - \tilde{\lambda}_{N+1} \tilde{p}_{N-2}^{(1)} = p_N - \lambda_{N+1} p_{N-2}^{(1)} = \mathcal{T}_N,$$

where the last equality follows from Corollary 3.1(b). Since $L^2(\tilde{\lambda}_N) < L^2$ we deduce that

$$\mathcal{T}_N^2 - L^2(\tilde{\lambda}_N) = \prod_{j=1}^{2N} (x - a_j(\tilde{\lambda}_N)) =: \tilde{H},$$

where $a_1(\tilde{\lambda}_N) < a_2(\tilde{\lambda}_N) < \cdots < a_{2N}(\tilde{\lambda}_N)$.

Applying Theorem 3.3 we get the orthogonality property given in (a) and (b) with

$$\tilde{\varepsilon}_k = \operatorname{sgn}[(\tilde{p}_N + \tilde{\lambda}_{N+1} \tilde{p}_{N-2}^{(1)}) / \sqrt{\tilde{H}}](w_k),$$

where w_k denotes the zeros of $\tilde{p}_{N-1} = p_{N-1}$. Since, by the first relation of (3.23) and the recurrence relation of $\tilde{p}_N$,

$$(\tilde{p}_N + \tilde{\lambda}_{N+1} \tilde{p}^{(1)}_{N-2})(\omega_k) = -\tilde{\lambda}_N p^{(1)}_{N-2}(w_k) \left[\frac{p_{N-2}(w_k)}{p^{(1)}_{N-2}(w_k)} - \frac{\tilde{\lambda}_{N+1}}{\tilde{\lambda}_N} \right]$$

and since, by Remark 3.4, $w_k \in [a_{2k}(\tilde{\lambda}_N), a_{2k+1}(\tilde{\lambda}_N)]$ for $k = 1, ..., l-1$, relation (3.22) follows by the interlacing property of the zeros of $p^{(1)}_{N-2}$ and p_{N-1} and from (1.2).

(c) First let us note that

$$\tilde{\lambda}_{N+1}/\tilde{\lambda}_N < (>) \, \lambda_{N+1}/\lambda_N \qquad \text{iff} \quad \tilde{\lambda}_N > (<) \, \lambda_N. \tag{3.24}$$

Taking into consideration the fact that

$$\varepsilon_\kappa = \operatorname{sgn} \left[\frac{p_{N-2}(w_{j_\kappa})}{p^{(1)}_{N-2}(w_{j_\kappa})} - \frac{\lambda_{N+1}}{\lambda_N} \right],$$

where w_{j_κ}, $\kappa = 1, ..., l-1$, denotes the zeros of $p_{N-1}/\mathscr{U}_{N-l}$, (c) is proved for the zeros of $p_{N-1}/\mathscr{U}_{N-l}$.

Concerning the zeros u_κ, $\kappa = 1, ..., N-l$, of $\mathscr{U}_{N-l}$ which are by Remark 3.4 zeros of p_{N-1} too, we observe that at u_κ

$$L^2 = \mathscr{T}^2_N = (p_N - \lambda_{N+1} p^{(1)}_{N-2})^2 = [(L^2/4\lambda_{N+1} p^{(1)}_{N-2}) + \lambda_{N+1} p_{N-2}]^2,$$

where we used the fact that by Lemma 3.1(b) at the zeros of p_{N-1}

$$-p_N p^{(1)}_{N-2} = L^2/4\lambda_{N+1}, \tag{3.25}$$

which gives

$$p^{(1)}_{N-2}(u_\kappa) = \pm L/2\lambda_{N+1}.$$

Inserting this in (3.25) and using the recurrence relation of p_N we have

$$(p_{N-2}/p^{(1)}_{N-2})(u_\kappa) = \lambda_{N+1}/\lambda_N$$

which proves in view of (3.24) and (3.22) the assertion. ∎

The following remark gives another representation of orthogonal polynomials with periodic recurrence coefficients (see [2] and also the methods used in [6, Theorem 7]).

Remark 3.6. Suppose that (p_n) satisfy (3.2) and that the recurrence coefficients of (p_n) satisfy

$$\alpha_n = \alpha_{N+n} \qquad \text{and} \qquad \lambda_{n+1} = \lambda_{N+n+1} \text{ for } n \in \mathbf{N}$$

and set

$$\mathscr{T}_N = p_N - \lambda_{N+1} p_{N-2}^{(1)} \qquad \text{and} \qquad L^2/4 = \prod_{j=2}^{N+1} \lambda_j.$$

Then for $0 \leqslant n \leqslant N-1$ and $k \in \mathbf{N}_0$

$$p_{kN+n} = p_n \widehat{U_k(\tilde{\mathscr{T}}_N)} + \left(\prod_{j=N+1}^{N+n+1} \lambda_j\right) p_{N-2-n}^{(n+1)} \widehat{U_{k-1}(\tilde{\mathscr{T}}_N)} \tag{3.26}$$

and for $0 \leqslant n \leqslant N-1$ and $k \in \mathbf{N} \setminus \{1\}$

$$2p_{kN+n} - \mathscr{T}_N p_{(k-1)N+n} = p_{kN+n} - (L^2/4)\, p_{(k-2)N+n} \tag{3.27}$$

and

$$2p_{N+n} - \mathscr{T}_N p_n = p_{N+n} + \left(\prod_{j=1}^{n+1} \lambda_{N+j}\right) p_{N-2-n}^{(n+1)}. \tag{3.28}$$

Proof. Relation (3.26) has been given in [2] and can be proved with the help of Lemma 3.1(c) and 3.1(b). Using (3.26) and

$$\widehat{U_k(\tilde{\mathscr{T}}_N)} = \mathscr{T}_N \widehat{U_{k-1}(\tilde{\mathscr{T}}_N)} - (L^2/4)\, \widehat{U_{k-2}(\tilde{\mathscr{T}}_N)}$$

we obtain (3.27) and (3.28). ∎

Example. Let $E_1 = [-1, +1]$. Then for each $N \in \mathbf{N}$, $\mathscr{T}_N = \hat{T}_N$ is a T-polynomial on $[-1, +1]$ and $\mathscr{U}_{N-1} = \hat{U}_{N-1}$. Furthermore $(p_n = \hat{U}_n)_{n \in \mathbf{N}}$ is orthogonal on $[-1, +1]$ with respect to $\sqrt{1-x^2}\, dx$, $\alpha_n = 0$, and $\lambda_{n+1} = 1/4$ for $n \in \mathbf{N}$, and $L^2 = 4^{-N+1}$.

Now let $(\tilde{p}_n)$ be the polynomials generated by the periodic recurrence coefficients $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$, $\lambda_2 = \cdots = \lambda_{N-1} = 1/4$, $\tilde{\lambda}_N$, $\tilde{\lambda}_{N+1} = 1/2 - \tilde{\lambda}_N$, where $\tilde{\lambda}_N \in (1/4, 1/2)$. Putting $L^2(\tilde{\lambda}_N) = 2^{-2N+3} \tilde{\lambda}_N (1 - 2\tilde{\lambda}_N)$ it follows from Theorem 3.4 that the polynomials $(\tilde{p}_n)$ are orthogonal on $E(L(\tilde{\lambda}_N)) := \{x : |\hat{T}_N(x)| \leqslant L(\tilde{\lambda}_N)\}$ with respect to $w(x) := \sqrt{L^2(\tilde{\lambda}_N) - \hat{T}_N^2} / |\hat{U}_{N-1}(x)|\, dx$.

In view of Remark 3.6 the polynomials $(\tilde{p}_n)$ can be represented in the form, $k \in \mathbf{N}_0$, $0 \leqslant n \leqslant N-1$

$$\tilde{p}_{kN+n} = \hat{U}_n \widehat{U_k(\tilde{\mathscr{T}}_N)} + 4^{-n} \tilde{\lambda}_{N+1} \hat{U}_{N-2-n} \widehat{U_{k-1}(\tilde{\mathscr{T}}_N)},$$

where $\tilde{\mathscr{T}}_N = \hat{T}_n / L(\tilde{\lambda}_N)$ and as usual $\hat{U}_{-1} = 0$.

If $\tilde{\lambda}_N \in (0, 1/4)$ then the polynomials $(\tilde{p}_n)$ are orthogonal on $E(L(\tilde{\lambda}_N))$ with respect to the distribution function $w(x)\, dx + \sum_{k=1}^{N-1} \mu_k \delta(x - w_k)$, where w_k are the zeros of $\hat{U}_{N-1}$ and $k \in \{1, ..., N-1\}$,

$$\mu_k = \frac{(-1)\sqrt{\hat{T}_N^2(w_k) - L^2(\tilde{\lambda}_N)}}{\hat{U}_{N-1}(w_k)} = \frac{(1 - w_k^2)\sqrt{1 - 8\tilde{\lambda}_N(1 - 2\tilde{\lambda}_N)}}{N}.$$

The above example (put $\tilde{\lambda}_N = c/2(1+c)$, $c \in \mathbf{R}^+$) is due to Ismail [8, Sect. 2] who derived it by direct methods.

Furthermore it follows by Theorem 3.4, Remark 3.6, and by simple calculations that the polynomials

$$q_{N+n} = \hat{U}_n \hat{T}_N + 2.4^{-n} \tilde{\lambda}_{N+1} \hat{U}_{N-2-n}$$

and

$$q_{kN+n} = \hat{U}_n \widehat{T_k(\tilde{\mathscr{T}}_N)} + 4^{-n} \tilde{\lambda}_{N+1} \hat{U}_{N-2-n} \widehat{T_{k-1}(\tilde{\mathscr{T}}_N)},$$

where $0 \leqslant n \leqslant N-1$ and $k \in \mathbf{N} \setminus \{1\}$, are orthogonal on $E(L(\tilde{\lambda}_N))$ with respect to $1/\sqrt{L^2(\tilde{\lambda}_N) - \hat{T}_N^2}\, |\hat{U}_{N-1}(x)|\, dx$ if $\tilde{\lambda}_N \in (0, 1/4)$.

In this section we have demonstrated that the recurrence coefficients of the polynomials orthogonal with respect to $\Psi_{R,\rho,\varepsilon}$ are periodic if and only if there exists a T-polynomial on E_l. Since we shall show in a forthcoming paper that for a given set of disjoint intervals $\tilde{E}_l$ and arbitrary $\varepsilon \in \mathbf{R}^+$ there exists a set of disjoint intervals E_l such that $\lambda(\tilde{E}_l \setminus E_l) < \varepsilon$, λ denotes the Lebesgue measure, and that there exists a T-polynomial on $\tilde{E}_l$, we get in a simple way that the recurrence coefficients of polynomials orthogonal with respect to $\Psi_{R,\rho,\varepsilon}$ are quasi-periodic in the limit. As already mentioned in the Introduction this quasi-periodic behaviour of the recurrence coefficients has been discovered by A. Magnus [10] using Abel functions.

4. Relations between the Recurrence Coefficients of the Polynomials Orthogonal with Respect to $\Psi_{R,\rho,\varepsilon}$ Resp. $\Psi_{R,\rho,-\varepsilon}$

In this section we show how the recurrence coefficients of the above mentioned orthogonal polynomials are related. In particular it is demonstrated that the recurrence coefficients of those polynomials which are orthogonal on E_l with respect to a distribution of the form $\sqrt{-R(x)/S(x)}\, dx$, where $r = l-1$ $(l+1)$ and $s = l+1$ $(l-1)$, are symmetric periodic if and only if there exists a T-polynomial on E_l. Polynomials orthogonal for such distributions also play an important role in L^1-approximation because the L^1-minimal polynomial on E_l can be represented with the help of such polynomials (see [13, Theorem 6]).

THEOREM 4.1. *Let $\mathscr{T}_N$ be a T-polynomial on E_l and let $\Psi_{R,\rho_v,\varepsilon}$ be definite. $(p_{n,\varepsilon})$ denotes the polynomials which are orthogonal with respect to $\Psi_{R,\rho_v,\varepsilon}$ and $(\alpha_{n,\varepsilon})$, $(\lambda_{n+1,\varepsilon})$ denote the recurrence coefficients of $(p_{n,\varepsilon})$. Then the following propositions hold for $k \in \mathbf{N}$:*

(a) *For $v + l - (r+1) < 2j < 2kN + v + l - (r+1)$*

$$p_{kN+l+v-(j+r+1),\,-\varepsilon} = (\mathscr{T}_{kN}\, p_{j,\varepsilon} - S\mathscr{U}_{kN-l}\, q_{i,\varepsilon}) / K_{j,\varepsilon}$$

and

$$q_{kN+l+v-(i+r+1),\,-\varepsilon}=(R\mathscr{U}_{kN-l}p_{j,\varepsilon}-\mathscr{T}_{kN}q_{i,\varepsilon})/K_{j,\varepsilon},$$

where $i=j+r-l$, $K_{j,\varepsilon}=A\prod_{\mu=1}^{j+1}\lambda_{\mu,\varepsilon}$, *and* $\rho_v=Ax^v+\cdots$.

(b) *For* $(v+l-r-1)/2<j<\min\{kN-2+v+l-r,\ kN-2+(v+l-r-1)/2\}$

$$\alpha_{kN+v+l-(j+1+r),\,-\varepsilon}=\alpha_{j+2,\varepsilon}$$

$$\lambda_{kN+v+l-(j+1+r),\,-\varepsilon}=\lambda_{j+3,\varepsilon}.$$

Proof. (a) Since by Lemma 2.1

$$\mathscr{T}_{kN}^2-H\mathscr{U}_{kN-l}^2=L_{kN}^2$$

and, by Theorem I.3,

$$Rp_{j,\varepsilon}^2-Sq_{i,\varepsilon}^2=\rho_v\,g_{(j)}$$

for $2j>v+l-r-1$, where $g_{(j)}\in\mathbf{P}_{l-1}$ and

$$(Rp_{j,\varepsilon})(w_k)=\varepsilon_k(\sqrt{H}\,q_{i,\varepsilon})(w_k)$$

at the zeros w_k of ρ_v, we get (for details see Lemma I.6) that

$$R(\mathscr{T}_{kN}p_{j,\varepsilon}-S\mathscr{U}_{kN-l}q_{i,\varepsilon})^2-S(R\mathscr{U}_{kN-l}p_{j,\varepsilon}-\mathscr{T}_{kN}q_{i,\varepsilon})^2=L_{kN}^2\rho_v\,g_{(j)},$$

where at the zeros w_k of ρ_v

$$R(\mathscr{T}_{kN}p_{j,\varepsilon}-S\mathscr{U}_{kN-l}q_{i,\varepsilon})(w_k)=-\varepsilon_k(\sqrt{H}(R\mathscr{U}_{kN-l}p_{j,\varepsilon}-\mathscr{T}_{kN}q_{i,\varepsilon}))(w_k)$$

and that the polynomial $\mathscr{T}_{kN}p_{j,\varepsilon}-S\mathscr{U}_{kN-l}q_{i,\varepsilon}$ resp. $R\mathscr{U}_{kN-l}p_{j,\varepsilon}-\mathscr{T}_{kN}q_{i,\varepsilon}$ is of degree $kN-j+v+\partial g_{(j)}-r$ resp. $kN-l-j+v+\partial g_{(j)}$ with leading coefficient $\tilde{K}_{j,\varepsilon}/2$, where $\tilde{K}_{j,\varepsilon}$ is the leading coefficient of $\rho_v\,g_{(j)}$; hence by Theorem I.3, $\tilde{K}_{j,\varepsilon}/2=A\prod_{\mu=1}^{j+1}\lambda_{\mu,\varepsilon}$. Part (a) follows now from Theorem I.1.

(b) Since $p_{j,\varepsilon}$ and $q_{i,\varepsilon}$ satisfy the same recurrence relation we get in conjunction with (a) that for $j\in\mathbf{N}_0$ with $v+l+3-r<2(j+2)<2kN+v+l-1$ and $kN+l+v-(j+r+3)\geqslant 0$

$$K_{j+2,\varepsilon}p_{kN+l+v-(j+r+3),\,-\varepsilon}=(x-\alpha_{j+2,\varepsilon})\,K_{j+1,\varepsilon}p_{kN+l+v-(j+r+2),\,-\varepsilon}-\lambda_{j+2,\varepsilon}K_{j,\varepsilon}p_{kN+l+v-(j+r+1),\,-\varepsilon}.$$

Using the fact that

$$K_{j+2,\varepsilon}=\lambda_{j+3,\varepsilon}K_{j+1,\varepsilon}$$

we obtain, dividing the above relation by $K_{j+1,\varepsilon}$ that

$$p_{kN+l+v-(j+r+1),-\varepsilon}$$
$$= (x-\alpha_{j+2,\varepsilon})\, p_{kN+l+v-(j+r+2),-\varepsilon} - \lambda_{j+3,\varepsilon}\, p_{kN+l+v-(j+r+3),-\varepsilon}$$

which proves part (b). ∎

COROLLARY 4.1. *Suppose that the assumptions of Theorem* 4.1 *are fulfilled and let* $R=H$ *and* $v=l-1$. *Then*

$$\alpha_{N-j,-\varepsilon}=\alpha_{j,\varepsilon} \quad \textit{for} \quad j=1,...,N-1, \quad \textit{and} \quad \alpha_{N,-\varepsilon}=\alpha_{N,\varepsilon},$$

and

$$\lambda_{N+1-j,-\varepsilon}=\lambda_{j,\varepsilon} \quad \textit{for} \quad j=2,...,N-1,$$
$$\lambda_{N,-\varepsilon}=\lambda_{N+1,\varepsilon}, \quad \textit{and} \quad \lambda_{N+1,-\varepsilon}=\lambda_{N,\varepsilon}.$$

Proof. Putting $k=1$ and $k=2$ in Theorem 4.1(b) and using the fact that by Theorem 3.1(c), $\alpha_{N+1,-\varepsilon}=\alpha_{1,-\varepsilon}$ the corollary follows. ∎

COROLLARY 4.2. *Suppose that the assumptions of Theorem* 4.1 *are fulfilled and let* $r=l-1$ *and* $\rho\equiv 1$. *Then*

$$\alpha_{N+2-j}=\alpha_j \quad \textit{for} \quad j=1,...,N+1$$
$$\lambda_{N+2-j}=\lambda_{j+1} \quad \textit{for} \quad j=2,...,N-1$$

and

$$\lambda_{N+1}=\lambda_{N+2}=\lambda_2/2.$$

Proof. In view of Theorem 4.1(b), since $v=0$ and thus there is no point measure, it remains only to show that

$$\alpha_{N+2-j}=\alpha_j \quad \text{for} \quad j=1,2 \quad \text{and} \quad \lambda_{N+2}=\lambda_{N+1}=\lambda_2/2.$$

By Corollary 2.1 we have that

$$p_N=\mathscr{T}_N. \tag{4.1}$$

Using the representation of p_{N-1} and p_{N-2} given in Theorem 4.1(a) and inserting these expressions in the recurrence relation of $p_N=\mathscr{T}_N$, we obtain that

$$\mathscr{T}_N\left[\left(\lambda_N\Big/\prod_{j=1}^{3}\lambda_j\right)p_2(x)-(x-\alpha_N)\left(p_1(x)\Big/\prod_{j=1}^{2}\lambda_j\right)+1\right]$$
$$=S\mathscr{U}_{N-l}\left[(x-\alpha_N)\,q_1(x)-\left(\lambda_N\Big/\prod_{j=1}^{3}\lambda_j\right)q_2(x)\right]. \tag{4.2}$$

Since $\mathcal{T}_N \neq 0$ at the zeros of $S\mathcal{U}_{N-l}$ it follows that the second term of the left side of (4.2) vanishes identically which implies that $\alpha_N = \alpha_2$.

The remaining relations are obtained similarily with the help of the recurrence relation of p_{N+1} using the representation of p_{N+1}, p_N, and p_{N-1} given in Theorem 3.1(a), (4.1), and Theorem 4.1(a), respectively. ∎

As already mentioned in Remark 2.1 polynomials orthogonal with respect to $\sqrt{-R(x)/S(x)}\,dx$ can also be fitted into that class of orthogonal polynomials investigated in [6]. But, to the best of our knowledge, Corollary 4.1, i.e., the symmetric periodic behaviour of the recurrence coefficients, can not be derived from the results of [6].

Next let us demonstrate that the converse of Corollary 4.2 holds also. We need

LEMMA 4.1. *Let m_0, m_1, $K \in \mathbf{N}_0$, and suppose that $m_0 + 1 \leqslant m_1 \leqslant K$. Then*

$$\alpha_{K+1-j} = \tilde{\alpha}_j \qquad \text{for} \quad j = m_0 + 1, ..., m_1$$
$$\lambda_{K+2-j} = \tilde{\lambda}_j \qquad \text{for} \quad j = m_0 + 2, ..., m_1$$

if and only if $\tilde{p}_j^{(m_0)} = p_j^{(K-m_0-j)}$ for $j = 1, ..., m_1 - m_0$.

Proof. Using the relations

$$\tilde{p}_j^{(m_0)} = (x - \tilde{\alpha}_{j+m_0})\,\tilde{p}_{j-1}^{(m_0)} - \tilde{\lambda}_{j+m_0}\,\tilde{p}_{j-2}^{(m_0)}$$

and, see Lemma 3.1(a),

$$p_j^{(K-m_0-j)} = (x - \alpha_{K+1-m_0-j})\,p_{j-1}^{(K+1-m_0-j)} - \lambda_{K+2-m_0-j}\,p_{j-2}^{(K+2-m_0-j)}$$

the assertion follows by induction. ∎

THEOREM 4.2. *Let $\alpha_n \in \mathbf{R}$, $\lambda_{n+1} \in \mathbf{R}^+$ for $n \in \mathbf{N}$, and suppose that for $k \in N$*

$$\alpha_{kN+j} = \alpha_j = \alpha_{N+2-j} \qquad \text{for} \quad j = 1, ..., N+1$$
$$\lambda_{kN+j} = \lambda_j = \lambda_{N+3-j} \qquad \text{for} \quad j = 3, ..., N$$

and

$$\lambda_{kN+1} = \lambda_{kN+2} = \lambda_2/2,$$

where $N \in \mathbf{N}$. Furthermore let p_n, $n \in \mathbf{N}$, denote that polynomials which are generated by the recurrence coefficients α_n and λ_{n+1}, $n \in \mathbf{N}$. Put

$$p_{N-1}^{(1)} = \mathcal{U}_{N-l}R \qquad \text{and} \qquad p_{N+1} - \lambda_{N+1}\,p_{N-1} = \mathcal{U}_{N-l}S, \tag{4.3}$$

where $\mathscr{U}_{N-l}$, R, S *are polynomials with leading coefficient one and* R *and* S *have no common zero. Then the following propositions hold*:

(a) $H := RS = \prod_{j=1}^{2l}(x-a_j)$, *where* $a_1 < a_2 < \cdots < a_{2l}$, *and either* $R(a_{2j}) = 0$ *or* $R(a_{2j+1}) = 0$ *for* $j = 1, ..., l-1$.

(b) *The polynomials* p_n, $n \in \mathbf{N}$, *are orthogonal on* $E_l = \bigcup_{j=1}^{l}[a_{2j-1}, a_{2j}]$ *with respect to the weight function* $\sqrt{-R/S}$.

(c) *The polynomials* $p_n^{(1)}$, $n \in \mathbf{N}$, *are orthogonal on* $E_l = \bigcup_{j=1}^{l}[a_{2j-1}, a_{2j}]$ *with respect to the weight function* $\sqrt{-S/R}$.

Proof. First let us note that

$$\alpha_{N+n+1} = \alpha_{n+1} \quad \text{for} \quad n \geqslant 1 \quad \text{and} \quad \lambda_{N+n+1} = \lambda_{n+1} \quad \text{for} \quad n \geqslant 2.$$

Further let $\mathscr{T}_N$, L, and ρ be defined as in Lemma 3.3 and Theorem 3.3. Using the relations

$$\alpha_{N+1} = \alpha_1 \quad \text{and} \quad \lambda_2 = 2\lambda_{N+1} = 2\lambda_{N+2} \tag{4.4}$$

and (Lemma 4.1)

$$p_{N-2}^{(1)} = p_{N-2}^{(2)} \tag{4.5}$$

we get with the help of Lemma 3.3(e) and (b) that

$$\mathscr{T}_N = p_N^{(1)} - \lambda_{N+2}\, p_{N-2}^{(2)} = (x-\alpha_1)\, p_{N-1}^{(1)} - \lambda_2\, p_{N-2}^{(2)} = p_N. \tag{4.6}$$

Since by Lemma 3.1(b)

$$L^2/2 = \prod_{j=2}^{n+1} \lambda_j = p_N^{(1)} p_N - p_{N+1}\, p_{N-1}^{(1)}$$
$$= \lambda_{N+1}(p_{N-1}^{(1)}\, p_{N-1} - p_N\, p_{N-2}^{(1)})$$

we derive with the help of (4.4), (4.5), and (4.6) that

$$p_N^2 - L^2 = p_{N-1}^{(1)}(p_{N+1} - \lambda_{N+1}\, p_{N-1}) \tag{4.7}$$

and thus, by (4.6) and (4.3), that

$$\mathscr{T}_N^2 - L^2 = H \mathscr{U}_{N-l}^2.$$

Now let $x_1 < x_2 < \cdots < x_N$ be the zeros of p_N. Then, using well known interlacing properties of the zeros of orthogonal polynomials,

$$\operatorname{sgn}(p_{N+1} - \lambda_{N+1}\, p_{N-1})(x_i) = -2\lambda_{N+1} \operatorname{sgn} p_{N-1}(x_i) = (-1)^{N+1-i}$$

and

$$\operatorname{sgn} p_{N-1}^{(1)}(x_i) = (-1)^{N-i}$$

for $i=1, ..., N$ from which it follows that there is exactly one zero of $p_{N-1}^{(1)}$ and $p_{N+1}-\lambda_{N+1}p_{N-1}$ in each interval (x_i, x_{i+1}), $i=1, ..., N$, which proves part (a).

From (3.16) we get by simple calculation in conjunction with (4.4) that

$$\rho\mathscr{U}_{N-l}=-\lambda_{N+1}(p_{N+1}-\lambda_{N+1}p_{N-1})$$

which implies by (4.7) that each zero of ρ is a zero of H. Thus we get from Theorem 3.3 that (p_n) is orthogonal with respect to $p_{N-1}^{(1)}/\mathscr{U}_{N-l}h\,dx$ which is the assertion for (p_n). The orthogonality property of $(p_n^{(1)})$ follows from Theorem I.3 combined with Theorem I.1 observing that by Theorem I.1(d), $q_m(x)=-p_{n-1}^{(1)}(x)$. ∎

5. Recurrence Relation for the Recurrence Coefficients

In this section we demonstrate how to get in a simple way (compared to [19]) recurrence relations for the recurrence coefficients if there exists a T-polynomial $\mathscr{T}_N$ on E_l, where $N>l$, and thus by Theorem 3.1, the period N of the recurrence coefficients is greater than the number l of the disjoint intervals. Assuming that the recurrence coefficients of the orthogonal polynomials are periodic Turchi *et al.* [19], using completely different methods, got the same recurrence laws for the recurrence coefficients as it should be in view of Theorem 3.3.

Notation. Let (p_k) be a sequence of orthogonal polynomials and let (α_k), (λ_{k+1}) be the recurrence coefficients of (p_k) with the property that $\lambda_{k+1}\neq 0$ for $k\in\mathbf{N}$. We set for $k, n\in\mathbf{N}_0$

$$p_k^{(n)}(x)=\sum_{j=0}^{k} A_j^{(k,n)}x^{k-j}, \qquad \text{where} \quad A_0^{(k,n)}=1. \tag{5.1}$$

Then we get from the recurrence relations (3.3) resp. Lemma 3.1(a) that for $n\in\mathbf{N}_0$, $k\in\mathbf{N}$,

$$A_j^{(k,n)}=A_j^{(k-1,n)}-\alpha_{k+n}A_{j-1}^{(k-1,n)}-\lambda_{k+n}A_{j-2}^{(k-2,n)} \qquad \text{for} \quad j=1, ..., k, \tag{5.2}$$

resp.

$$A_j^{(k,n)}=A_j^{(k-1,n+1)}-\alpha_{n+1}A_{j-1}^{(k-1,n+1)}-\lambda_{n+2}A_{j-2}^{(k-2,n+2)} \qquad \text{for} \quad j=1, ..., k, \tag{5.3}$$

where

$$A_j^{(k,n)}:=0 \qquad \text{for} \quad j\in\{k+1, k+2, ...,\}\cup\{-1\}.$$

Now let us demonstrate how to get the recurrence relations. Suppose that there exists a T-polynomial $\mathscr{T}_N$, $N \geqslant l$ on E_l, and that $\Psi_{R,\rho_\nu,\varepsilon}$ is definite. Let (p_n) be orthogonal with respect to $\Psi_{R,\rho_\nu,\varepsilon}$ having recurrence coefficients (α_n), (λ_{n+1}). We put

$$\mathscr{T}_N(x) = \sum_{j=0}^{N} \tau_j x^{N-j} \qquad \text{and} \qquad \mathscr{U}_{N-l}(x) = \sum_{j=0}^{N-l} u_j x^{N-l-j},$$

and for $n \geqslant n_0$

$$\hat{g}_{(n)}(x) = \sum_{j=0}^{l-1} G_j^{(n)} x^{l-1-j} \qquad \text{and} \qquad \hat{f}_{(n+1)}(x) = \sum_{j=0}^{l} F_j^{(n+1)} x^{l-j},$$

where $\mathscr{U}_{N-l}$ is defined in (2.1) and $\hat{g}_{(n)}$ and $\hat{f}_{(n+1)}$ are defined in Corollary 3.2. Observing that by Corollary 3.1(b) for $n \geqslant n_0$

$$p_N^{(n+1)} + \lambda_{N+n+2} p_{N-2}^{(n+2)} = \mathscr{T}_N + 2\lambda_{N+n+2} p_{N-2}^{(n+2)}$$

we get from Corollary 3.1(b) resp. Corollary 3.2 that the following fundamental relations hold for $n \geqslant n_0$

$$\tau_j = A_j^{(N,n+1)} - \lambda_{N+n+2} A_{j-2}^{(N-2,n+2)} \qquad \text{for} \quad j = 1, ..., N, \tag{5.4}$$

$$\sum_{\mu=0}^{j} u_{j-\mu} G_\mu^{(n)} = A_j^{(N-1,n+1)} \qquad \text{for} \quad j = 1, ..., N-1 \tag{5.5}$$

$$\sum_{\mu=0}^{j} u_{j-\mu} F_\mu^{(n+1)} = \tau_j + 2\lambda_{N+n+2} A_{j-2}^{(N-2,n+2)} \qquad \text{for} \quad j = 1, ..., N, \tag{5.6}$$

where $G_\mu^{(n)} := 0$ and $F_{\mu+1}^{(n+1)} := 0$ for $\mu \geqslant l$.

Since by (5.3) and (5.4) for $n \geqslant n_0$

$$\begin{aligned} A_j^{(N-1,n+1)} = \tau_j &+ \alpha_{N+n+1} A_{j-1}^{(N-1,n+1)} + \lambda_{N+n+1} A_{j-2}^{(N-2,n+1)} \\ &+ \lambda_{N+n+2} A_{j-2}^{(N-2,n+2)} \qquad \text{for} \quad j = 1, ..., N-1, \end{aligned} \tag{5.7}$$

the expression on the right side of (5.5) and (5.6) can be expressed in terms of τ_k's and u_k's and in terms of the recurrence coefficients (α_k), (λ_{k+1}) by using successively (5.7) and the following relations which can be derived from (5.2) resp. (5.3).

$$A_j^{(N-2,n+1)} = A_j^{(N-1,n+1)} + \alpha_{N+n} A_{j-1}^{(N-2,n+1)} + \lambda_{N+n} A_{j-2}^{(N-3,n+1)} \tag{5.8}$$

and

$$\begin{aligned} A_j^{(N-k-1,n+k+1)} = A_j^{(N-k,n+k)} &+ \alpha_{n+k+1} A_{j-1}^{(N-k-1,n+k+1)} \\ &+ \lambda_{n+k+2} A_{j-2}^{(N-k-2,n+k+2)} \end{aligned} \tag{5.9}$$

$k \in \{0, ..., N-1\}$, in conjunction with

$$\alpha_{n+k+1} = \alpha_{N+n+k+1} \quad \text{and} \quad \lambda_{n+k+2} = \lambda_{N+n+k+2} \quad \text{for} \quad n \geqslant n_0. \tag{5.10}$$

Thus we obtain from the first $l-1$ resp. l equations of (5.5) resp. (5.6) the coefficients $G_j^{(n)}$ resp. $F_j^{(n+1)}$ of $\hat{g}_{(n)}$ resp. $\hat{f}_{(n+1)}$ in terms of τ_k's and u_k's and in terms of the recurrence coefficients. Finally we get from the lth equation of (5.5) resp. (5.6) two (nonlinear) recurrence relations for the recurrence coefficients. Further relations for the recurrence coefficients can be obtained by considering (5.5) resp. (5.6) for $j > l$ resp. $j > l+1$.

EXAMPLES. Suppose that there exists a T-polynomial $\mathscr{T}_N$ on E_l and that $\Psi_{R,\rho_\nu,\varepsilon}$ is definite. Let (p_n) be orthogonal with respect to $\Psi_{R,\rho_\nu,\varepsilon}$ on E_l and let (α_n), (λ_{n+1}) be the recurrence coefficients of (p_n) Furthermore let n_0 be defined as in Theorem 3.1. Then the recurrence coefficients have period N, by Theorem 3.1, and satisfy the following recurrence relations:

(a) $l=2$. Put $C_1 = u_1 - \tau_1$. Then

$$\lambda_{n+2} + \lambda_{n+1} + \alpha_{n+1}(\alpha_{n+1} - C_1) = (u_2 - \tau_2) + u_1(\tau_1 - u_1) =: C_2 \quad \text{for} \quad n \geqslant n_0 + 1 \tag{5.11}$$

$$2\lambda_{n+2}(\alpha_{n+2} + \alpha_{n+1} - C_1) = u_3 - \tau_3 + u_1(\tau_2 - u_2) + (u_2 - u_1^2)(\tau_1 - u_1) =: C_3 \quad \text{for} \quad n \geqslant n_0 + 1. \tag{5.12}$$

If $\alpha_{N+n_0+1} = \alpha_{n_0+1}$, then (5.12) holds also for $n = n_0$.

(b) $l=3$. Let C_1, C_2, and C_3 be defined as above. Then

$$\lambda_{n+2}(\alpha_{n+2} + 2\alpha_{n+1} - C_1) + \lambda_{n+1}(\alpha_n + 2\alpha_{n+1} - C_1) + \alpha_{n+1}[\alpha_{n+1}(\alpha_{n+1} - C_1) - C_2] = C_3 \quad \text{for} \quad n \geqslant n_0 + 2 \tag{5.13}$$

and

$$\begin{aligned} 2\lambda_{n+2}[\lambda_{n+3} &+ \lambda_{n+2} + \lambda_{n+1} + \alpha_{n+1}(\alpha_{n+1} - C_1) \\ &+ \alpha_{n+2}(\alpha_{n+2} - C_1) + \alpha_{n+2}\alpha_{n+1} - C_2] \\ &= u_4 - \tau_4 + u_1(\tau_3 - u_3) + (u_2 - u_1^2)(\tau_2 - u_2) \\ &\quad - [u_3 - u_2 u_1 - u_1(u_2 - u_1^2)]\, C_1 =: C_4 \end{aligned} \tag{5.14}$$

for $n \geqslant n_0 + 1$, where (5.13) holds also for $n = n_0 + 1$, if $\alpha_{n_0+1} = \alpha_{N+n_0+1}$.

Proof. (a) From (5.5) resp. (5.6) it follows immediately that

$$u_2 - u_1^2 = A_2^{(N-1,n+1)} - u_1 A_1^{(N-1,n+1)}$$

resp.

$$u_3 + (u_2 - u_1^2) F_1^{(n+1)} - u_2 u_1 = \tau_3 - u_1 \tau_2 + 2\lambda_{N+n+2}(A_1^{(N-2,n+2)} - u_1)$$

which gives with the help of (5.7), (5.9), and (5.10) the assertion.

(b) From (5.5) resp. (5.6) it follows by straightforward calculation that

$$\begin{aligned} &u_3 - u_1 u_2 + u_1(u_1^2 - u_2) \\ &\quad = A_3^{(N-1,n+1)} - u_1 A_2^{(N-1,n+1)} + (u_1^2 - u_2) A_1^{(N-1,n+1)} \end{aligned}$$

resp.

$$C_4 = 2\lambda_{N+n+2}[A_2^{(N-2,n+2)} - u_1 A_1^{(N-2,n+2)} - (u_2 - u_1^2)]$$

from which with the help of (5.7), (5.8), (5.9), and (5.10) the assertion follows. ∎

Note that the recurrence relations do not depend on ρ_ν. In order to calculate those recurrence coefficients which are not determined by the recurrence relations for the recurrence coefficients, i.e., those of low index, one has to calculate the "first" moments

$$m_j^{(R,\rho,\varepsilon)} = \Psi_{R,\rho,\varepsilon}(x^j).$$

(If $\rho = 1$ and $\varepsilon = (1, 1, ..., 1)$ we write $m_j^{(R)}$.) This can be done with the help of relations (3.6) and (3.1) or by the following method which is often simpler.

Let $u \in \mathbf{P}_{r-l}$ be such that

$$(R/\sqrt{H} - u)(z) = 0(z^{-1}).$$

Observing that by (3.6)

$$\int_{E_l} \frac{R(x)}{z-x} \frac{dx}{h(x)} + u(z) = (R/\sqrt{H})(z) \tag{5.15}$$

and that for sufficiently large $|z|$

$$y(z) := (R/\sqrt{H})(z) = z^{r-l} \sum_{j=0}^{\infty} d_j z^{-j}$$

satisfies the differential equation

$$2Hy' = y(R'S - S'R) \tag{5.16}$$

we get, equating the coefficients in (5.16), the coefficients of u and the moments $m_j^{(R)} = d_{r-l+1+j}$, $j \in \mathbf{N}_0$.

Now let us suppose that all zeros of ρ_ν are simple and let us put

$$\rho_\nu(x) = \prod_{k=1}^{\nu} (x - w_k) = \sum_{j=0}^{\nu} B_j x^j$$

and for $k = 1, ..., \nu$,

$$-\rho_{\nu,k}(x) = \frac{\rho_\nu(x)}{w_k - x} = \sum_{j=0}^{\nu-1} B_{k,j} x^j.$$

Then the moments $m_j^{(R,\rho,\varepsilon)}$, $j = 0, ..., \nu - 1$, can be calculated by the system of linear equations

$$\sum_{j=0}^{\nu-1} B_{k,j} m_j^{(R,\rho,\varepsilon)} = \varepsilon_k (R/\sqrt{H})(w_k) - u(w_k) \qquad \text{for} \quad k = 1, ..., \nu, \tag{5.17}$$

and the moments of higher index with the help of the relation

$$\sum_{j=0}^{\nu} B_j m_{j+k}^{(R,\rho,\varepsilon)} = m_k^{(R)} \qquad \text{for} \quad k \in \mathbf{N}_0, \tag{5.18}$$

where (5.17) resp. (5.18) are obtained by considering $\Psi_{R,\rho_\nu,\varepsilon}(-\rho_{\nu,k})$ and using (5.15) resp. by considering $\Psi_{R,\rho_\nu,\varepsilon}(x^k \rho_\nu)$.

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